Rectifiability of sets and measures

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State of affairs I

Let μ be a Radon measure in $\mathbb{R}^m$ such that

$$0 < \theta^n(\mu, a) = \lim_{r \to 0} \frac{\mu(B(a,r))}{r^n} < \infty \text{ for } \mu - a.e\ a \in \mathbb{R}^m$$

Then for μ-a.e. $a \in \mathbb{R}^m$ every $\nu \in \text{Tan}(\mu, a)$ is n-uniform with $0 \in \text{spt}\,\nu$.

Thus to understand the infinitesimal structure μ we need to understand the structure of n-uniform measures.

Uniformly distributed measures - Kirchheim-Preiss

- A measure μ on $\mathbb{R}^m$ is uniformly distributed if there is a function $f_\mu : (0,\infty) \to [0,\infty]$ such that
  - $\mu(B(x,r)) = f_\mu(r)$ for all $x \in \operatorname{spt}\mu$ and all $r > 0$.
  - $f_\mu(r) < \infty$ for some r.

- For μ uniformly distributed in $\mathbb{R}^m$, $x \in \mathbb{R}^m$ and $0 < s \leq r < \infty$

$$\mu(B(x,r)) \leq 5^m \left(\frac{r}{s}\right)^m f_\mu(s). \qquad (*)$$

Cover $B(x,r) \subset \cup_{i=1}^N B(z_i, s/2)$ with $|z_i - z_j| \geq s/2$.
Since $\{B(z_i, s/4)\}_{i=1}^N$ are disjoint then $N\left(\frac{s}{4}\right)^m \leq \left(\frac{5}{4}r\right)^m$.

$$\mu(B(x,r)) \leq \sum_{i=1}^{N} \mu(B(z_i, s/2)).$$

Consider 2 cases: $\mu(B(z_i, s/2)) = 0$ or $\mu(B(z_i, s/2)) > 0$. If $\mu(B(z_i, s/2)) > 0$ there is $z \in \operatorname{spt}\mu \cap B(z_i, s/2)$, and $B(z_i, s/2) \subset B(z, s)$.

Uniformly distributed measures - Kirchheim-Preiss

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  - $f_\mu(r) < \infty$ for some r.

- For μ uniformly distributed in $\mathbb{R}^m$, $x \in \mathbb{R}^m$ and $0 < s \leq r < \infty$

$$\mu(B(x,r)) \leq 5^m \left(\frac{r}{s}\right)^m f_\mu(s).$$

- For μ uniformly distributed in $\mathbb{R}^m$ $(*)$
  - $f_\mu(r) < \infty$ for all $r > 0$
  - $\displaystyle\int e^{-s|x-z|^2}\, d\mu(z) = \int_0^1 \mu(B(x, \sqrt{-\frac{\ln r}{s}}))\, dr$ converges for $s > 0$, $x \in \mathbb{R}^m$.
  - $\displaystyle\int e^{-s|x-z|^2}\, d\mu(z) = \int e^{-s|y-z|^2}\, d\mu(z)$ for all $x, y \in \operatorname{spt}\mu$ and $s > 0$.

Uniformly distributed measures - Kirchheim-Preiss

Let μ be uniformly distributed in $\mathbb{R}^m$, $x_0 \in \operatorname{spt}\mu$, for $s > 0$ and $x \in \mathbb{R}^m$

- $F(x,s) = \displaystyle\int \left(e^{-s|x-z|^2} - e^{-s|x_0-z|^2} \right) d\mu(z)$ is well defined and independent of x_0.
- $x \in \operatorname{spt}\mu$ iff $F(x,s) = 0$ for all $s > 0$.
- If $x \notin \operatorname{spt}\mu$ there is $s_x > 0$ so that $s > s_x$, $F(x,s) < 0$ $(*)$.
- $\operatorname{spt}\mu$ is a real analytic variety.
- There are $n \in \{0, 1, \cdots, m\}$, $c > 0$ and $G \subset \mathbb{R}^m$ an open set such that
  - $G \cap \operatorname{spt}\mu$ is an n-dimensional analytic submanifold of $\mathbb{R}^m$.
  - $\mu(\mathbb{R}^m \setminus G) = \mathcal{H}^n(\mathbb{R}^m \setminus G) = 0$.
  - $\mathbb{R}^m \setminus G$ countable union of analytic submanifolds of $\mathbb{R}^m$ of dimension less than n.
  - $\mu(A) = c\mathcal{H}^n(A \cap G \cap \operatorname{spt}\mu) = c\mathcal{H}^n(A \cap \operatorname{spt}\mu)$ for $A \subset \mathbb{R}^m$ Borel.

State of affairs II

Let μ be a Radon measure in $\mathbb{R}^m$ such that

$$0 < \theta^n(\mu, a) = \lim_{r \to 0} \frac{\mu(B(a,r))}{r^n} < \infty \text{ for } \mu - a.e\ a \in \mathbb{R}^m$$

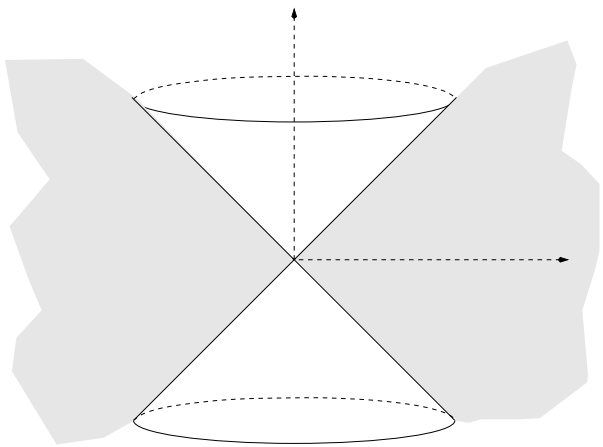
Then:

- For μ-a.e. $a \in \mathbb{R}^m$ every $\nu \in \text{Tan}(\mu, a)$ is n-uniform thus uniformly distributed.
- For $x \in G \cap \text{spt}\,\nu$ (n-analytic submanifold) if $\lambda \in \text{Tan}(\nu, x)$ then there exists $c > 0$ such that $\lambda = c\mathcal{H}^n \llcorner V_x$ where $V_x = T_x\text{spt}\,\nu - x$.
- Since tangents to tangents are tangents $\lambda \in \text{Tan}(\mu, a)$.
- $\lambda \in \mathcal{F}_{n,m} = \mathcal{F} = \{c\mathcal{H}^n \llcorner V : c > 0,\ V \in G(m,n)\}$, i.e. λ is flat.
- For μ-a.e. $a \in \mathbb{R}^m$, $\text{Tan}(\mu, a) \cap \mathcal{F} \neq \emptyset$.

Are all n-uniform measures flat? How large can the singular set be?

Theorem If ν is n-uniform in $\mathbb{R}^{n+1}$, $\Sigma = \operatorname{spt}\nu$ then $\nu = c\mathcal{H}^n \llcorner \Sigma$ and modulo translation and rotation

- **Preiss** For $n = 1, 2$ $\Sigma = \mathbb{R}^n \times \{0\}$
- **Kowalski-Preiss** For $n \geq 3$,
  - $\Sigma = \mathbb{R}^n \times \{0\}$, or
  - $\Sigma = \{(x_1, x_2, x_3, x_4, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : x_4^2 = x_1^2 + x_2^2 + x_3^2\}$.



Recent work of D. Nimer addresses the question in higher codimensions.

How do we prove rectifiability of μ?

Theorem (Preiss) Let μ be a Radon measure in $\mathbb{R}^m$ with

$$0 < \theta_*^n(\mu, a) \leq \theta^{n,*}(\mu, a) < \infty \quad \mu - a.e.\ a \in \mathbb{R}^m.$$

Then the following are equivalent:

- μ is n-rectifiable
- $\mu - a.e\ a \in \mathbb{R}^m$ there is V_a an n-dimensional space in $\mathbb{R}^m$ so that

$$\text{Tan}(\mu, a) = \{c\mathcal{H}^n \llcorner V_a : 0 < c < \infty\}$$

- $\mu - a.e.\ a \in \mathbb{R}^m$, $\text{Tan}(\mu, a) \subset \mathcal{F}$.

Does $\text{Tan}(\mu, a) \cap \mathcal{F} \neq \emptyset \implies \text{Tan}(\mu, a) \subset \mathcal{F}$ for μ-a.e. $a \in \mathbb{R}^m$?

Metric on the space of Radon measures in $\mathbb{R}^m$

For $r \in (0, \infty)$, let

$$\mathcal{L}(r) = \{f : \mathbb{R}^m \to [0, \infty), \operatorname{spt} f \subset B_r \text{ and } \operatorname{Lip} f \leq 1\}.$$

For Radon measures Φ and Ψ on $\mathbb{R}^m$ set

$$F_r(\Phi, \Psi) = \sup \left\{ \left| \int f \, d\Phi - \int f \, d\Psi \right| ; f \in \mathcal{L}(r) \right\}.$$

$$F_r(\Phi) = F_r(\Phi, 0) = \int (r - |z|)^+ \, d\Phi(z). \qquad (*)$$

- If μ, μ_i $(i \in \mathbb{N})$ Radon measures in $\mathbb{R}^m$, then

$$\mu_i \rightharpoonup \mu \text{ iff } \lim_{i \to \infty} F_r(\mu_i, \mu) = 0 \quad \forall r > 0.$$

Cones of measures

- Let $\mathcal{R}$ be the space of Radon measures in $\mathbb{R}^m$. For $\Phi, \Psi \in \mathcal{R}$ define

$$d(\Phi, \Psi) = \sum_{i=1}^{\infty} 2^{-i} \min\{1, F_i(\Phi, \Psi)\}.$$

  $(\mathcal{R}, d)$ is a complete separable metric space. $(*)$
- If μ, μ_i $(i \in \mathbb{N})$ Radon measures in $\mathbb{R}^m$, then

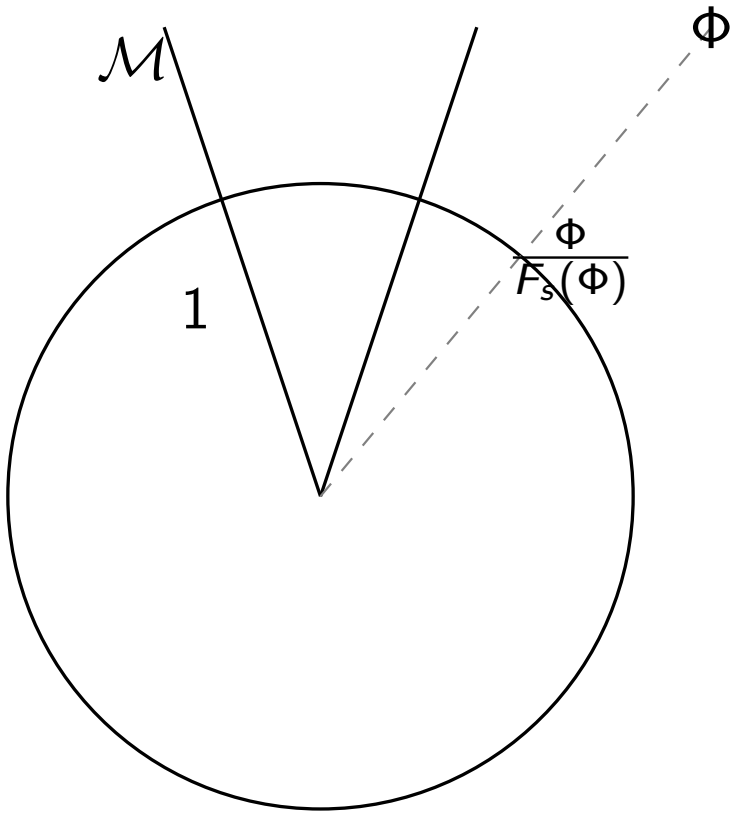
$$\mu_i \rightharpoonup \mu \text{ iff } \lim_{i \to \infty} d(\mu_i, \mu) = 0.$$

- $\mathcal{M} \subset \mathcal{R}$, $0 \notin \mathcal{M}$ is a *cone* if $c\Psi \in \mathcal{M}$ for all $\Psi \in \mathcal{M}$ and $c > 0$.
- A cone $\mathcal{M}$ is a *d-cone* if $T_{0,r}[\Psi] \in \mathcal{M}$ for all $\Psi \in \mathcal{M}$.
- For $s > 0$ the *s-distance* between a d-cone $\mathcal{M}$ and $\Phi \in \mathcal{R}$ is

$$d_s(\Phi, \mathcal{M}) = \inf \left\{ F_s\left(\frac{\Phi}{F_s(\Phi)}, \Psi\right) : \Psi \in \mathcal{M} \text{ and } F_s(\Psi) = 1 \right\}$$

  if $F_s(\Phi) \neq 0$. If $F_s(\Phi) = 0$ set $d_s(\Phi, \mathcal{M}) = 1$.

A picture to illustrate $d_s(\Phi, \mathcal{M})$



$$\{\Psi \in \mathcal{R} : F_s(\Psi) = 1\}$$

$$d_s(\Phi, \mathcal{M}) = \inf \left\{ F_s(\tfrac{\Phi}{F_s(\Phi)}, \Psi) : \Psi \in \mathcal{M} \text{ and } F_s(\Psi) = 1 \right\}$$

Additional properties

- If $\mathcal{M}$ d-cone and $\Phi \in \mathcal{R}$
  - $d_s(\Phi, \mathcal{M}) \leq 1$ $(*)$
  - $d_s(\Phi, \mathcal{M}) = d_1(T_{0,s}[\Phi], \mathcal{M})$
- If $\mu_i \rightharpoonup \mu$ and $F_s(\mu) > 0$ then $d_s(\mu, \mathcal{M}) = \lim_{i \to \infty} d_s(\mu_i, \mathcal{M})$.
- If μ is a non-zero Radon measure $\mathrm{Tan}(\mu, a)$ is a d-cone.
  If $\nu \in \mathrm{Tan}(\mu, a)$, $T_{0,r}[\nu] \in \mathrm{Tan}(\mu, a)$.
- If μ is a non-zero Radon measure $\{\nu \in \mathrm{Tan}(\mu, a) : F_1(\nu) = 1\}$ is closed under weak convergence (i.e. in the topology induced by d).
- The basis of a d-cone $\mathcal{M}$ is the set

$$\{\Psi \in \mathcal{M} : F_1(\Psi) = 1\}.$$

Compact basis

- Let $\mathcal{M}$ be a d-cone in $\mathcal{R}$ with a closed basis then $\mathcal{M}$ has a compact basis iff there exists $\kappa > 1$ such that

$$\Psi(B(0,2r)) \leq \kappa \Psi(B(0,r))$$

for all $\Psi \in \mathcal{M}$ and $r > 0$,i.e. the doubling constant is uniform on $\mathcal{M}$.

- Let $\mu \in \mathcal{R}$, $a \in \operatorname{spt} \mu$ if

$$c_a = \limsup_{r \to 0} \frac{\mu(B(a,2r))}{\mu(B(a,r))} < \infty,$$

then $\operatorname{Tan}(\mu, a)$ has a compact basis. $(*)$

Connectivity properties of $\mathrm{Tan}(\mu, a)$ - Preiss

Let $\mathcal{M}$ and $\mathcal{F}$ be d-cones in $\mathcal{R}$. Assume that $\mathcal{F} \subset \mathcal{M}$, $\mathcal{F}$ relatively closed in $\mathcal{M}$ and $\mathcal{M}$ has a compact basis. Suppose that there exists $\epsilon_0 > 0$ such that if

$$(*) \qquad d_r(\Phi, \mathcal{F}) < \epsilon_0 \quad \forall r \geq r_0 > 0 \text{ then } \Phi \in \mathcal{F}.$$

Then for a Radon measure μ and $a \in \operatorname{spt} \mu$ if

$$\mathrm{Tan}(\mu, a) \subset \mathcal{M} \text{ and } \mathrm{Tan}(\mu, a) \cap \mathcal{F} \neq \emptyset \text{ then } \mathrm{Tan}(\mu, a) \subset \mathcal{F}$$

State of affairs III

Let μ be a Radon measure in $\mathbb{R}^m$ such that

$$0 < \theta^n(\mu, a) = \lim_{r \to 0} \frac{\mu(B(a,r))}{r^n} < \infty \text{ for } \mu - a.e\ a \in \mathbb{R}^m$$

Then for μ-a.e. $a \in \mathbb{R}^m$:

- $\mathcal{F} \subset \mathcal{M}$, where $\mathcal{F}$ is the set of n-flat measures in $\mathbb{R}^m$ and $\mathcal{M}$ is the set of n-uniform measures in $\mathbb{R}^m$ with 0 in their support.
- $\mathcal{F}$ is closed in $\mathcal{M}$, which is a d-cone with compact basis.
- $\text{Tan}(\mu, a) \subset \mathcal{M}$.
- $\text{Tan}(\mu, a) \cap \mathcal{F} \neq \emptyset$.
- If $(*)$ holds then $\text{Tan}(\mu, a) \subset \mathcal{F} \implies \mu$ n-rectifiable

What does $(*)$ really mean?

Let Φ be n-uniform. Recall

$$d_r(\Phi, \mathcal{F} = \inf \left\{ F_r(\frac{\Phi}{F_r(\Phi)}, \Psi) : \Psi \in \mathcal{F} \text{ and } F_r(\Psi) = 1 \right\}.$$

$$\begin{aligned} F_r(\Phi) &= \int (r - |y|)^+ \, d\Phi = \int \Phi(\{y : (r - |y|)^+ > t\}) \, dt \\ &= \int_0^r \Phi(\{y : |y| < r - t\})) \, dt = \int_0^r \Phi(B_{r-t}) \, dt \\ &= \int_0^r c(r - t)^n \, dt = c \frac{r^{n+1}}{n+1} \end{aligned}$$

- $d_r(\Phi, \mathcal{F}) = d_r(c\Phi, \mathcal{F})$ for $c = \omega_n$ then $F_r(\Phi) = \frac{\omega_n r^{n+1}}{n+1}$.
- If $\Psi = c\mathcal{H}^n \llcorner V$, $F_r(\Psi) = c\frac{\omega_n r^{n+1}}{n+1} = 1$ implies $c = \left(\frac{\omega_n r^{n+1}}{n+1} \right)^{-1}$.

$$d_r(\Phi, \mathcal{F}) = \inf_{V \in G(m,n)} \frac{n+1}{\omega_n r^{n+1}} \sup_{f \in \mathcal{L}(r)} \left| \int f \, d\Phi - \int f \, d\mathcal{H}^n \llcorner V \right|$$

Note that for

$$f(x) = \text{dist}^{\,2}(x, V) \frac{(r - |x|)^+}{r^2} \in \mathcal{L}(r).$$

$$\begin{aligned} \frac{n+1}{\omega_n r^{n+1}} \left| \int f \, d\Phi - \int f \, d\mathcal{H}^n \llcorner V \right| &\geq \frac{n+1}{\omega_n r^{n+1}} \int f \, d\Phi \\ &\geq \frac{n+1}{\omega_n r^{n+1}} \int_{B_{r/2}} \text{dist}^{\,2}(x, V) \frac{r}{2r^2} \, d\Phi \\ &\geq \frac{n+1}{2\omega_n r^{n+2}} \int_{B_{r/2}} \text{dist}^{\,2}(x, V) \, d\Phi \end{aligned}$$

If $(*)$ $d_r(\Phi, \mathcal{F}) < \epsilon_0$ for $r \geq r_0 > 0$, then

$$\inf_{V \in G(m,n)} \frac{n+1}{\omega_n r^{n+2}} \int_{B_{r/2}} \text{dist}^{\,2}(x, V) \, d\Phi < 2\epsilon_0.$$

The condition

$$d_r(\Phi, \mathcal{F}) < \epsilon_0 \quad \forall r \geq r_0$$

implies that there exists $\epsilon_1 > 0$ such that

$$\inf_{V \in G(m,n)} \frac{1}{\omega_n r^{n+2}} \int_{B_r} \text{dist}^{\,2}(x, V)\, d\Phi < \epsilon_1 \text{ for } r \geq r_0.$$

Thus for $r \geq r_0$ there exists $V_r \in G(m, n)$ such that

$$(\spadesuit) \qquad \frac{1}{\omega_n r^{n+2}} \int_{B_r} \text{dist}^{\,2}(x, V_r)\, d\Phi \leq \epsilon_1$$

To prove $(*)$ we need to show that if Φ n-uniform is "close to flat at infinity" in L^2 as in $(\spadesuit)$ then Φ is flat.

(♠) yields geometric information

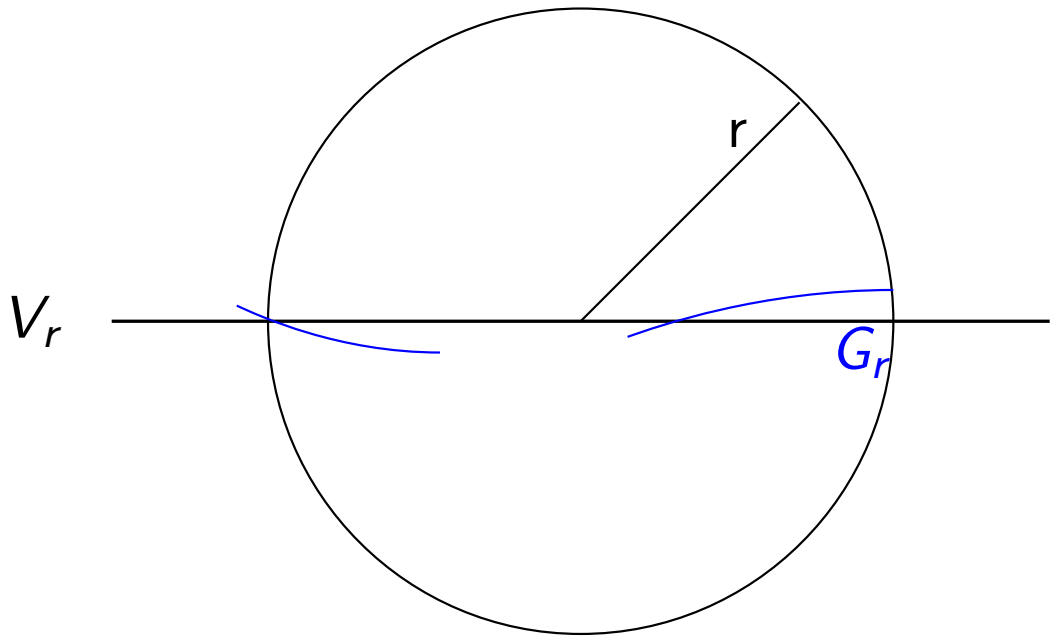
There is a large good subset G_r of $\operatorname{spt}\Phi$ in B_r which is close to V_r

By Chebyshev's inequality if (♠) holds then

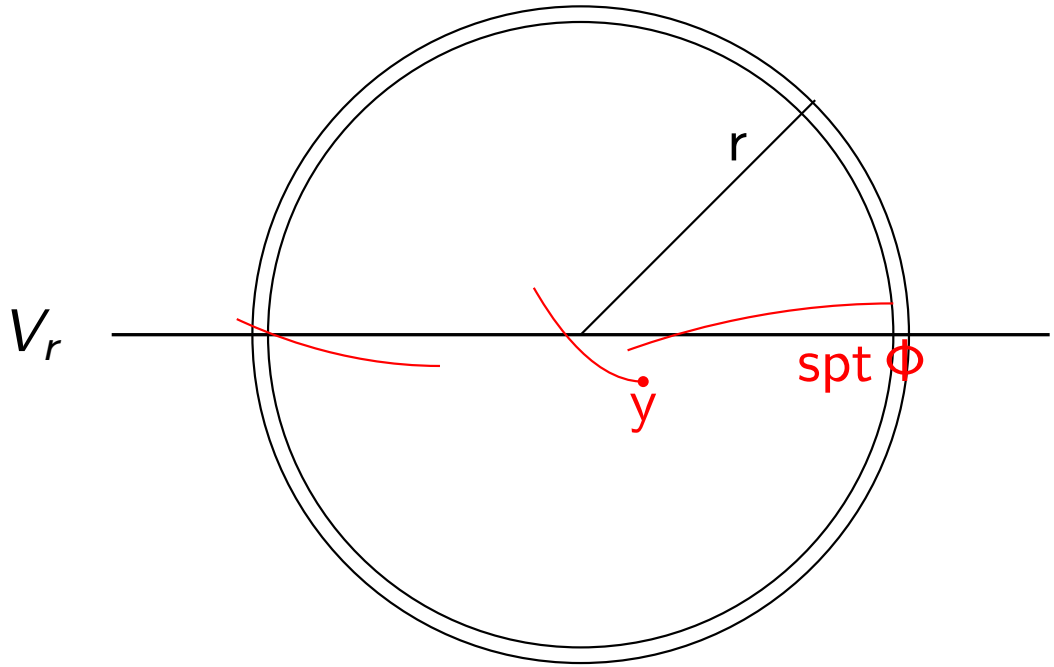
$$G_r = \{x \in \operatorname{spt}\Phi \cap B_r : \operatorname{dist}(x, V_r) \leq \epsilon_1^{1/4} r\}$$

satisfies

$$\Phi(B_r \setminus G_r) \leq \epsilon_1^{1/2} \Phi(B_r).$$



The small subset of $\operatorname{spt}\Phi$ in $B_r \setminus G_r$ is also close to V_r



Let $r_1 = r(1 - 2\epsilon_1^{1/2n})$. If $y \in (B_{r_1} \setminus G_r) \cap \operatorname{spt}\Phi$,

$\rho = \min\{\operatorname{dist}(y, G_r), \operatorname{dist}(y, \partial B_r)\} > 0$ then $B_\rho(y) \subset B_r \setminus G_r$

$$\omega_n \rho^n = \Phi(B_\rho(y)) \leq \Phi(B_r \setminus G_r) \leq \epsilon_1^{1/2} \Phi(B_r) = \omega_n \epsilon_1^{1/2} r^n.$$

Thus $\rho = \operatorname{dist}(y, G_r) \leq \epsilon_1^{1/2n} r$. For $y \in (B_{r(1-2\epsilon_1^{1/2n})} \setminus G_r) \cap \operatorname{spt}\Phi$,

$$\operatorname{dist}(y, V_r) \leq \operatorname{dist}(y, G_r) + \operatorname{dist}(G_r, V_r) \leq \epsilon_1^{1/2n} r.$$

Summary

If $d_r(\Phi, \mathcal{F}) < \epsilon_0$ for $r \geq r_0 > 0$ there is V_r such that

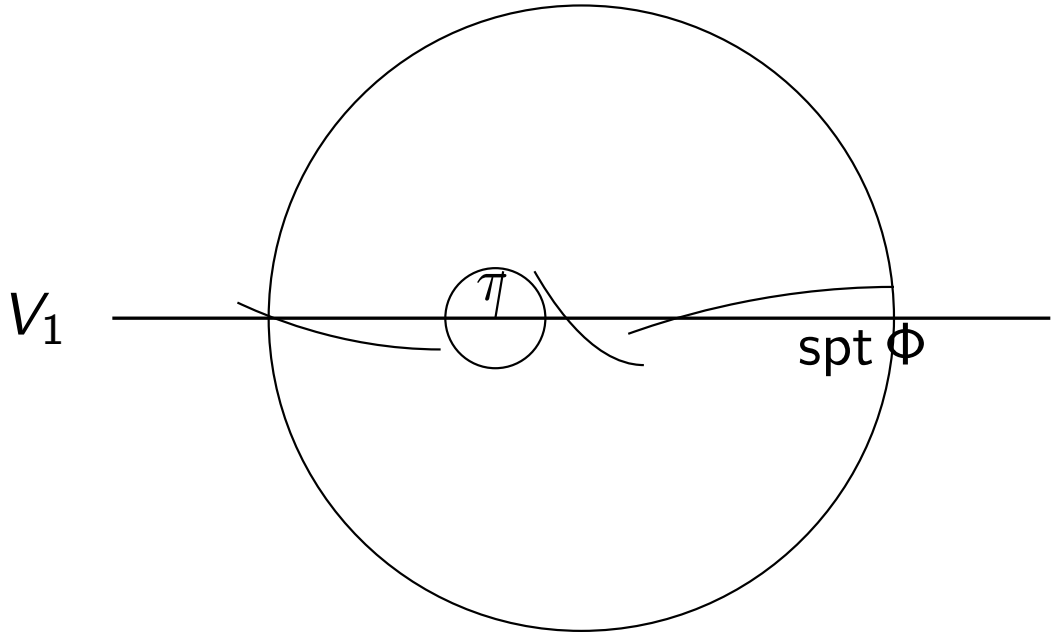
$$(\spadesuit) \qquad \frac{1}{\omega_n r^{n+2}} \int_{B_r} \text{dist}^{\,2}(x, V_r)\, d\Phi \leq \epsilon_1$$

with $\epsilon_1 = C(n)\epsilon_0$.
In this case for $y \in B_{r(1-2\epsilon_1^{1/2n})} \cap \text{spt}\,\Phi$, $\text{dist}\,(y, V_r) \leq \epsilon_1^{1/2n} r$.

$\text{spt}\,\Phi$ close to V_r in L^2 sense in B_r then $\text{spt}\,\Phi$ close V_r in L^∞ sense in $B_{r(1-2\epsilon_1^{1/2n})}$

(♠) yields additional geometric information for Φ n-uniform

Not only is spt Φ close to V_r but V_r is also close to spt Φ.



Can there be holes of size τ?
NO if ϵ_1 is small enough.

(♠) yields additional geometric information for Φ n-uniform

Lemma: Given $\tau > 0$ there exists $\epsilon_1 = \epsilon_1(\tau, n, m) > 0$ such that if Φ is n-uniform in $\mathbb{R}^m$ with $0 \in \operatorname{spt}\Phi$, $\Phi(B_1) = 1$ and for some $V \in G(m, n)$

$$\int_{B_1} \operatorname{dist}^2(x, V)\, d\Phi < \epsilon_1,$$

then for all $z \in V \cap B_1$

$$B(z, \tau) \cap \operatorname{spt}\Phi \neq \emptyset.$$