Rectifiability of sets and measures

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Tangent measures

A non-zero Radon measure ν is a tangent measure to μ at $a \in \mathbb{R}^m$ if there exist sequences $\{r_i\}_i$, $r_i \to 0$ and $\{c_i\}_i$, $c_i > 0$ such that

$$c_i T_{a,r_i}[\mu] \rightharpoonup \nu \text{ as } i \to \infty,$$

i.e. $\forall \varphi \in C_c(\mathbb{R}^m)$

$$\lim_{i\to\infty} c_i \int \varphi(\frac{x-a}{r_i})\, d\mu(x) = \int \varphi \, d\nu.$$

The set of all tangent measures to μ at a is denoted by

$$\text{Tan}(\mu, a).$$

Existence

- Preiss (1987) For any Radon measure μ, $\mathrm{Tan}(\mu, a) \neq \emptyset$, μ- a.e. $a \in \mathbb{R}^m$.
- Let μ be a Radon measure in $\mathbb{R}^m$, if for $a \in \mathbb{R}^m$

$$c_a = \limsup_{r \to 0} \frac{\mu(B(a, 2r))}{\mu(B(a, r))} < \infty \quad (\mu \text{ asymptotically doubling at } a)$$

then every sequence $\{r_i\}_i$, $r_i \to 0$ contains a subsequence $\{r_{i_k}\}_{i_k}$ such that

$$\mu_{a, r_{i_k}} = \frac{1}{\mu(B(a, r_{i_k}))} T_{a, r_{i_k}}[\mu] \rightharpoonup \nu \neq 0 \quad \text{and} \quad \nu \in \mathrm{Tan}(\mu, a).$$

Moreover any tangent measure is obtained this way up to a constant.

Proof

Since

$$1 \leq c_a = \inf_{s>0} \sup_{0<r<s} \frac{\mu(B(a,2r))}{\mu(B(a,r))} < \infty$$

there is $s > 0$ such that

$$\sup_{0<r<s} \frac{\mu(B(a,2r))}{\mu(B(a,r))} \leq 2c_a.$$

Let $\mu_i = \frac{1}{\mu(B(a,r_i))} T_{a,r_i}[\mu]$. Given a compact set $K \subset \mathbb{R}^m$, $K \subset B(0,2^\ell)$ for some ℓ. Then for i large enough

$$\mu_i(K) \leq \mu_i(B(0,2^\ell)) = \frac{\mu(B(a,2^\ell r_i))}{\mu(B(a,r_i))} \leq (2c_a)^\ell.$$

Thus $\sup_i \mu_i(K) < \infty$. By compactness there exists a subsequence such that $\mu_{i_k} \rightharpoonup \nu$ where ν is a Radon measure.

Proof

- $\nu \neq 0$ and $0 \in \operatorname{spt} \nu$: for $j \in \mathbb{N}$

$$\begin{aligned} \nu(B(0,2^{-j})) \geq \nu(\overline{B(0,2^{-j-1})}) &\geq \limsup_{i_k \to \infty} \mu_{i_k}(\overline{B(0,2^{-j-1})}) \\ &\geq \limsup_{i_k \to \infty} \frac{\mu(B(a,2^{-j-1}r_i))}{\mu(B(a,r_i))} \\ &\geq (2c_a)^{-j-1} > 0. \end{aligned}$$

- Let $\lambda \in \operatorname{Tan}(\mu, a)$ then $c_i T_{a,r_i}[\mu] \rightharpoonup \lambda$ for some $c_i > 0$ and $r_i \to 0$. Note that

$$c_i T_{a,r_i}[\mu](B(0,1)) = c_i \mu(B(a,r_i)) \sim 1 \text{ uniformly on } i.$$

Passing to a subsequence $c_{i_k}\mu(B(a,r_{i_k})) \to c_0 > 0$ and

$$\frac{c_0}{\mu(B(a,r_{i_k}))} T_{a,r_{i_k}}[\mu] \rightharpoonup \lambda.$$

Remarks

If for $s > 0$

$$\begin{aligned} \theta^s_*(\mu, a) &= \liminf_{r\to 0} \frac{\mu(B(a,r))}{r^s} > 0 \\ \theta^{*,s}(\mu, a) &= \limsup_{r\to 0} \frac{\mu(B(a,r))}{r^s} < \infty, \end{aligned}$$

then μ asymptotically doubling at a. Note

$$\frac{\mu(B(a,2r))}{\mu(B(a,r))} = 2^s \frac{\mu(B(a,2r))}{(2r)^s} \cdot \frac{r^s}{\mu(B(a,r))}.$$

$$\limsup_{r\to 0} \frac{\mu(B(a,2r))}{\mu(B(a,r))} \leq 2^s \frac{\theta^{*,s}(\mu,a)}{\theta^s_*(\mu,a)}.$$

For $\lambda \in \mathrm{Tan}(\mu, a)$ there exists $c_0 > 0$ and $r_i \to 0$ such that

$$\frac{c_0}{\mu(B(a,r_i))} T_{a,r_i}[\mu] \rightharpoonup \lambda \text{ and } \frac{\mu(B(a,r_i))}{r_i^s} \to c_1.$$

Hence

$$c_0 c_1^{-1} r_i^{-s} T_{a,r_i}[\mu] \rightharpoonup \lambda$$

Densities and tangent measures

Theorem (Preiss) Let $s > 0$, μ a Radon measure in $\mathbb{R}^m$. Let

$$A = \{a \in \mathbb{R}^m : 0 < \theta^s_*(\mu, a) \leq \theta^{s,*}(\mu, a) < \infty\}$$

then for μ-a.e. $a \in A$ and every $\nu \in \mathrm{Tan}(\mu, a)$ there is $c > 0$ such that for all $x \in \mathrm{spt}\,\nu$ and $r > 0$

$$tcr^s \leq \nu(B(x, r)) \leq cr^s$$

where

$$t = t(a) = \frac{\theta^s_*(\mu, a)}{\theta^{s,*}(\mu, a)}.$$

Proof

- $A = \cup_{j=1}^{\infty} A_j$ such that $\theta_*^s(\mu, \cdot)$ and $\theta^{s,*}(\mu, \cdot)$ oscillate very little.
- Given $\epsilon > 0$ decompose each A_i into a countable collection so that there are $B_i \subset A$ and $\tau_i, c_i, r_i > 0$ such that $\mu(A \setminus \cup_i^{\infty} B_i) = 0$ and for $a \in B_i$ and $0 < r < r_i$

$$\tau_i \leq t(a) \leq \tau_i + \epsilon \quad \& \quad \tau_i c_i r^s \leq \mu(B(a, r_i)) \quad \leq c_i r^s.$$

- If for all $a \in B$ and $r \leq r_0$

$$\tau c r^s \leq \mu(B(a, r)) \leq c r^s.$$

Then if a is a μ-density point of B, and $\nu \in \mathrm{Tan}(\mu, a)$ with $\nu = \lim_{r_i \to 0} r_i^{-s} T_{a, r_i}[\mu]$ then for all $x \in \operatorname{spt} \nu$ and $r > 0$

$$\tau c r^s \leq \nu(B(x, r)) \leq c r^s.$$

Densities and tangent measures

Definition: Let $s > 0$. A Radon measure ν in $\mathbb{R}^m$ is s-uniform if there is $c > 0$ such that for all $x \in \operatorname{spt} \nu$ and $r > 0$

$$\nu(B(x, r)) = cr^s.$$

Example: $\mathcal{H}^m$ in $\mathbb{R}^m$. Are there others?
Find an example of a measure ν in $\mathbb{R}^3$ such that $\nu(B(x, r)) = \pi r^2$ for all $x \in \operatorname{spt} \nu$ and $0 < r < 1$.

Corollary (Preiss) Let $s > 0$, μ a Radon measure in $\mathbb{R}^m$. Let

$$A = \{a \in \mathbb{R}^m : 0 < \theta^s(\mu, a) < \infty\}$$

then for μ-a.e. $a \in A$ and every $\nu \in \operatorname{Tan}(\mu, a)$, ν is s-uniform with $0 \in \operatorname{spt} \nu$.

Densities

Theorem (Marstrand) (1954) Let $s > 0$, μ a Radon measure in $\mathbb{R}^m$. Let

$$A = \{a \in \mathbb{R}^m : 0 < \theta^s(\mu, a) < \infty\}.$$

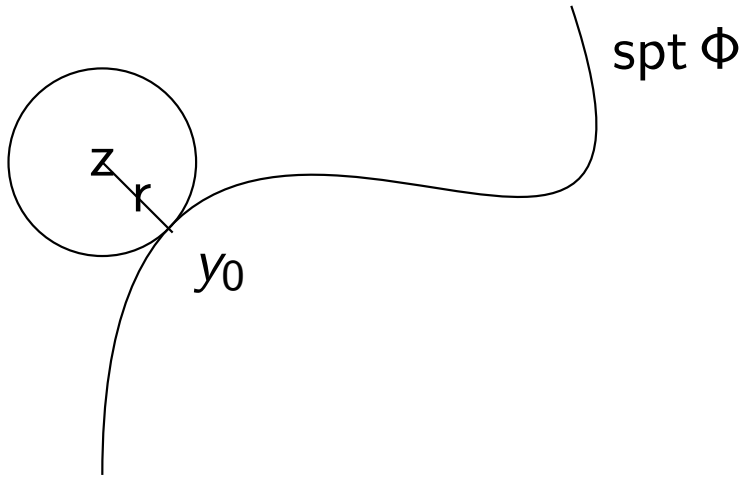
If $\mu(A) > 0$ then $s \in \mathbb{N}$.

Proof:

- For μ-a.e. $a \in A$ and every $\Phi \in \mathrm{Tan}(\mu, a)$, Φ is s-uniform with $0 \in \mathrm{spt}\,\Phi$.
- The theorem reduces to show that if Φ is a non-zero s-uniform measure in $\mathbb{R}^m$ then $s \in \mathbb{N}$.

Contradiction argument

- Let $\Phi \neq 0$ be s-uniform in $\mathbb{R}^m$ with $s \notin \mathbb{N}$ and $0 \in \operatorname{spt}\Phi$.
- Let m be the smallest dimension where this happens. By a dimension reduction argument there exists an s-uniform measure in $\mathbb{R}^{m-1}$.
- Since $s < m$, $\operatorname{spt}\Phi \neq \mathbb{R}^m$ (Check this using a covering argument). Moreover since $\operatorname{spt}\Phi$ is a closed set there exist $z \in \mathbb{R}^m \backslash \operatorname{spt}\Phi$ and $r > 0$ such that $B(z,r) \cap \operatorname{spt}\Phi = \emptyset$ but $y_0 \in \partial B(z,r) \cap \operatorname{spt}\Phi$.



Contradiction argument

- There exists $\nu \in \mathrm{Tan}(\Phi, y_0)$, ν s-uniform.
- The blow up of $\overline{B(z,r)}$ at y_0 is the half space $\{x : \langle x, e\rangle \leq 0\}$ where $e = \frac{y_0 - z}{|y_0 - z|}$ and $\mathrm{spt}\,\nu \subset \{x : \langle x, e\rangle \geq 0\}$.
- Let $B_r = B_r(0)$ consider the center of mass of ν on B_r is

$$b_r = \frac{1}{\nu(B_r)} \int_{B_r} z \, d\nu.$$

The center of mass - First moment estimate

- If $b_r = 0$ for all $r > 0$ then

$$0 = \langle b_r, e\rangle = \frac{1}{\nu(B_r)} \int_{B_r} \langle z, e\rangle \, d\nu \geq 0 \implies \langle z, e\rangle = 0 \text{ for all } z \in \operatorname{spt} \nu.$$

  Then $\operatorname{spt} \nu \subset \langle e\rangle^{\perp}$ an $(m-1)$-dimensional space: contradiction.

- If $b_r \neq 0$ for some $r > 0$ we estimate

$$2\langle b_r, y\rangle = \frac{2}{\nu(B_r)} \int_{B_r} \langle x, y\rangle \, d\nu(x)$$

  for $y \in \operatorname{spt} \nu$.

- **First moment estimate:** For $y \in \operatorname{spt} \nu \cap B_{r/2}$ since $0 \in \operatorname{spt} \nu$ we have

$$|2\langle b_r, y\rangle| \leq C(r)|y|^2$$

Quadratic estimate

For $y \in \operatorname{spt}\nu$ since $0 \in \operatorname{spt}\nu$

- $2\langle x, y\rangle = (r^2 - |x-y|^2) - (r^2 - |x|^2) + |y|^2$
  - $\dfrac{2}{\nu(B_r)} \displaystyle\int_{B_r} \langle x, y\rangle \, d\nu(x) =$
    $|y|^2 + \dfrac{1}{\nu(B_r)} \displaystyle\int_{B_r} (r^2 - |x-y|^2) \, d\nu(x) - \dfrac{1}{\nu(B_r)} \displaystyle\int_{B_r} (r^2 - |x|^2) \, d\nu(x)$
- $\displaystyle\int_{B_r} (r^2 - |x|^2) \, d\nu(x) = \int_{B_r(y)} (r^2 - |x-y|^2) \, d\nu(x)$

A property of uniformly distributed measures

For $y \in \operatorname{spt}\nu$ since $0 \in \operatorname{spt}\nu$

$$\begin{aligned}
\int_{B_r(y)} (r^2 - |x-y|^2)\, d\nu(x) &= \int_0^{r^2} \nu(\{x : (r^2 - |x-y|^2 > t\})\, dt \\
&= \int_0^{r^2} \nu(B(y, \sqrt{t - r^2}))\, dt \\
&= \int_0^{r^2} \nu(B(0, \sqrt{t - r^2}))\, dt \\
&= \int_{B_r} (r^2 - |x|^2)\, d\nu(x)
\end{aligned}$$

Quadratic estimate

For $y \in \operatorname{spt}\nu \cap B_{r/2}$ since $0 \in \operatorname{spt}\nu$

- $2\langle x, y\rangle = |y|^2 + (r^2 - |x-y|^2) - (r^2 - |x|^2)$
- $\displaystyle\int_{B_r} (r^2 - |x|^2)\, d\nu(x) = \int_{B_r(y)} (r^2 - |x-y|^2)\, d\nu(x)$
- $\displaystyle|2\langle b_r, y\rangle| \leq |y|^2 + \frac{1}{\nu(B_r)} \left| \int_{(B_r(y)\setminus B_r)\cup(B_r\setminus B_r(y))} (r^2 - |x-y|^2)\, d\nu(x) \right|$
  - For $x \in (B_r(y)\setminus B_r) \subset (B_{r+|y|}\setminus B_r)$, $0 \leq r^2 - |x-y|^2 \leq 3r|y|$
  - For $x \in (B_r\setminus B_r(y)) \subset (B_r\setminus B_{r-|y|})$, $0 \leq |x-y|^2 - r^2 \leq 3r|y|$
- $\displaystyle|2\langle b_r, y\rangle| \leq |y|^2 + \frac{3r|y|}{\nu(B_r)}[\nu(B_{r+|y|}\setminus B_r) + \nu(B_r\setminus B_{r-|y|})]$

Quadratic estimate

For $y \in \operatorname{spt}\nu \cap B_{r/2}$ since $0 \in \operatorname{spt}\nu$

- $2\langle x, y\rangle = |y|^2 + (r^2 - |x-y|^2) - (r^2 - |x|^2)$
- $|2\langle b_r, y\rangle| \leq |y|^2 + \dfrac{3r|y|}{\nu(B_r)}[\nu(B_{r+|y|}\setminus B_r) + \nu(B_r \setminus B_{r-|y|})]$
- $|2\langle b_r, y\rangle| \leq |y|^2 + \dfrac{3r|y|}{\nu(B_r)}[\nu(B_{r+|y|}\setminus B_{r-|y|})]$
- $|2\langle b_r, y\rangle| \leq |y|^2 + \dfrac{6r|y|}{\nu(B_r)}[(r+|y|)^s - (r-|y|)^s]$

Quadratic estimate

For $y \in \operatorname{spt}\nu \cap B_{r/2}$ since $0 \in \operatorname{spt}\nu$

- $2\langle x, y\rangle = |y|^2 + (r^2 - |x-y|^2) - (r^2 - |x|^2)$
- $\displaystyle\int_{B_r} (r^2 - |x|^2)\, d\nu(x) = \int_{B(y,r)} (r^2 - |x-y|^2)\, d\nu(x)$
- $\displaystyle|2\langle b_r, y\rangle| \leq |y|^2 + \frac{3r|y|}{\nu(B_r)}[(r+|y|)^s - (r-|y|)^s]$
- $\displaystyle|2\langle b_r, y\rangle| \leq |y|^2 + 6\frac{r|y|}{r^s}[(r+|y|)^s - (r-|y|)^s]$
- $|2\langle b_r, y\rangle| \leq C(r)|y|^2$

Another blow up

- For $y \in \operatorname{spt}\nu \cap B_{r/2}$ since $0 \in \operatorname{spt}\nu$ we have

$$|2\langle b_r, y\rangle| \leq C(r)|y|^2 \Longrightarrow |2\langle b_r, \frac{y}{r_i}\rangle| \leq C(r) r_i |\frac{y}{r_i}|^2$$

- Let $\lambda \in \operatorname{Tan}(\nu, 0)$ then λ is s-uniform. For $\eta > 0$ and $R > 1$

$$\lambda(\{y \in B_R : |\langle y, b_r\rangle > \eta|y|\}) = 0.$$

$$\begin{aligned}
\lambda(G) &\leq \liminf_{i\to\infty} r_i^{-s} T_{0,r_i}[\nu](G) \\
&\leq \liminf_{i\to\infty} r_i^{-s} \nu(\{y \in B_{Rr_i} : |\langle y, b_r\rangle > \eta|y|\}) \\
&\leq \liminf_{i\to\infty} r_i^{-s} \nu(\{y \in B_{Rr_i} : \eta|y| \leq C|y|^2\}) \\
&\leq \liminf_{i\to\infty} r_i^{-s} \nu(\{y \in B_{Rr_i} : \eta \leq C|y|\}) = 0
\end{aligned}$$

Thus $\operatorname{spt}\lambda \subset \{z \in \mathbb{R}^m : \langle z, b_r\rangle = 0\}$: contradiction.

Summary of the contradiction argument

- Since the s density exists μ-a.e, for μ-a.e. $a \in A$ and every $\Phi \in \mathrm{Tan}(\mu, a)$, Φ is s-uniform with $0 \in \operatorname{spt} \Phi$.
- Let m be the smallest dimension where there is $\Phi \neq 0$ s-uniform in $\mathbb{R}^m$ with $s \notin \mathbb{N}$.
- By blow-up argument there exists ν s-uniform with support in a half plane.
- Study of the first moment at 0 and possibly an additional blow-up yields an s-uniform measure supported in an $(m-1)$-dimensional space.
- Contradiction

Take away

- Let μ be a Radon measure such that the density $\theta^s(\mu, a) \in (0, \infty)$ μ-a.e in $\mathbb{R}^m$ then μ-a.e. $a \in \mathbb{R}^m$ every $\nu \in \mathrm{Tan}(\mu, a)$, ν is s-uniform with $0 \in \mathrm{spt}\, \nu$.
- The moments play an important role in understanding the structure of n-uniform measures.
- Studying the tangent measures to a tangent measure can be useful.

Tangents to tangents are tangents

Theorem (Preiss): Let μ be a Radon measure in $\mathbb{R}^m$, μ- a.e. $a \in \mathbb{R}^m$ all $\nu \in \mathrm{Tan}(\mu, a)$ satisfy

- $T_{x,\rho}[\nu] \in \mathrm{Tan}(\mu, a)$ for all $x \in \mathrm{spt}\,\nu$ and all $\rho > 0$.
- $\mathrm{Tan}(\nu, x) \subset \mathrm{Tan}(\mu, a)$ for all $x \in \mathrm{spt}\,\nu$.

Remark: If μ is asymptotically doubling at a, then for all $\nu \in \mathrm{Tan}(\mu, a)$, $0 \in \mathrm{spt}\,\nu$ and in particular for all $\rho > 0$, $T_{0,\rho}[\nu] \in \mathrm{Tan}(\mu, a)$.