Rectifiability of sets and measures

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References

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Geometry of Measures

In the 1920's Besicovitch studied sets with locally finite "length" in the plane.
What is the length of a set?

The basic question: do the infinitesimal properties of the "length" of a set E in the plane yield geometric information about E itself?
What are the infinitesimal properties of the 'length" of a set E? What sort of geometric information can one expect?

This simple question marks the beginning of the study of the geometry of measures and the associated field known as Geometric Measure Theory!

$\mathcal{H}^s$: s-dimensional Hausdorff measure in $\mathbb{R}^m$

Let $A \subset \mathbb{R}^m$, $0 \leq s < \infty$, $0 < \delta \leq \infty$

$$\mathcal{H}^s_\delta(A) := \inf \left\{ \sum_{j=1}^{\infty} (\operatorname{diam} C_j)^s | A \subset \bigcup_{j=1}^{\infty} C_j, \ \operatorname{diam} C_j \leq \delta \right\}$$

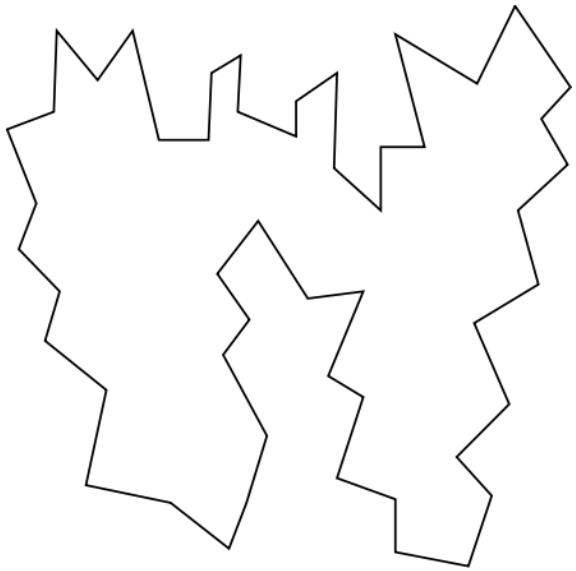
$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

Properties:

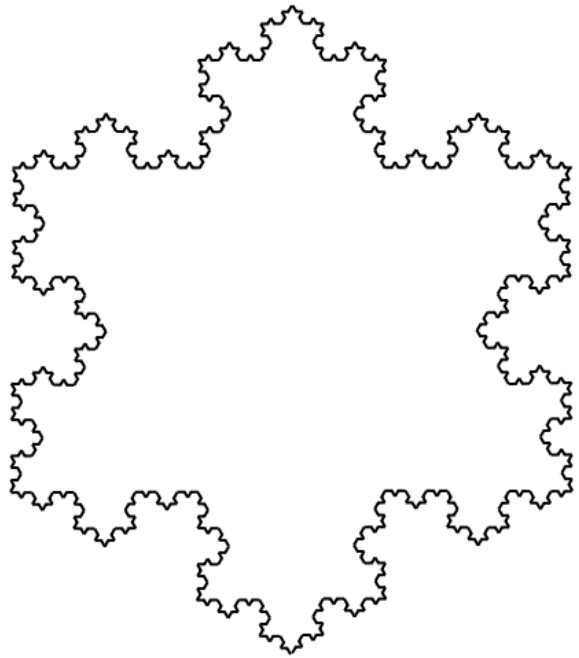
- $\mathcal{H}^s$ is a Borel measure.
- $\mathcal{H}^m = \mathcal{L}^m$ (m-Lebesgue measure) in $\mathbb{R}^m$.
- $\mathcal{H}^s \equiv 0$ for $s > m$.
- $\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A)$ for $\lambda > 0$.

$$\dim A := \inf \{ 0 \leq s < \infty | \, \mathcal{H}^s(A) = 0 \}$$

Examples



Hausdorff dimension 1



Hausdorff dimension $\frac{\log 4}{\log 3}$

Structure Theorem

Besicovitch (1929-1939) Suppose $X \subset \mathbb{R}^2$ and $0 < \mathcal{H}^1(X) < \infty$. Then

$$X = Y \cup Z$$

- $Y \subset \bigcup_{i=1}^{\infty} \Gamma_i$, where Γ_i is a Lipschitz image of $\mathbb{R}$.
- $\mathcal{H}^1(\pi_L(Z)) = 0$ for almost every line $L \subset \mathbb{R}^2$. Here π_L denotes the orthogonal projection of $\mathbb{R}^2$ onto L.

Federer (1947) Suppose $X \subset \mathbb{R}^m$, $n \in \mathbb{N}$ and $0 < \mathcal{H}^n(X) < \infty$. Then

$$X = Y \cup Z$$

- $Y \subset \bigcup_{i=1}^{\infty} \Sigma_i$, where Σ_i is a Lipschitz image of $\mathbb{R}^n$
- $\mathcal{H}^n(\pi_L(Z)) = 0$ for almost every n-dimensional plane $L \subset \mathbb{R}^m$. Here π_L denotes the orthogonal projection of $\mathbb{R}^m$ onto L.

Rectifiable sets

- A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz if there exists $M > 0$ such that for $x, y \in \mathbb{R}^n$

$$|f(x) - f(y)| \leq M|x - y|$$

  $\Sigma = f(\mathbb{R}^n)$ is a Lipschitz image of $\mathbb{R}^n$.

- $E \subset \mathbb{R}^m$ is n-rectifiable, $n \in \{1, \cdots, m\}$ if there exists a family $\{\Sigma_i\}_i$ of Lipschitz images of $\mathbb{R}^n$ such that

$$\mathcal{H}^n \left(E \setminus \bigcup_{i=1}^{\infty} \Sigma_i \right) = 0,$$

  i.e.

$$E \subset \left(\bigcup_{i=1}^{\infty} \Sigma_i \right) \cup \Sigma_0$$

  with $\mathcal{H}^n(\Sigma_0) = 0$.

Remarks

- By Whitney's extension theorem $E \subset \mathbb{R}^m$ is n-rectifiable iff there exists a family $\{\widetilde{\Sigma}_i\}_i$ n-dimensional smooth sub-manifolds and a set $\widetilde{\Sigma}_0$ such that

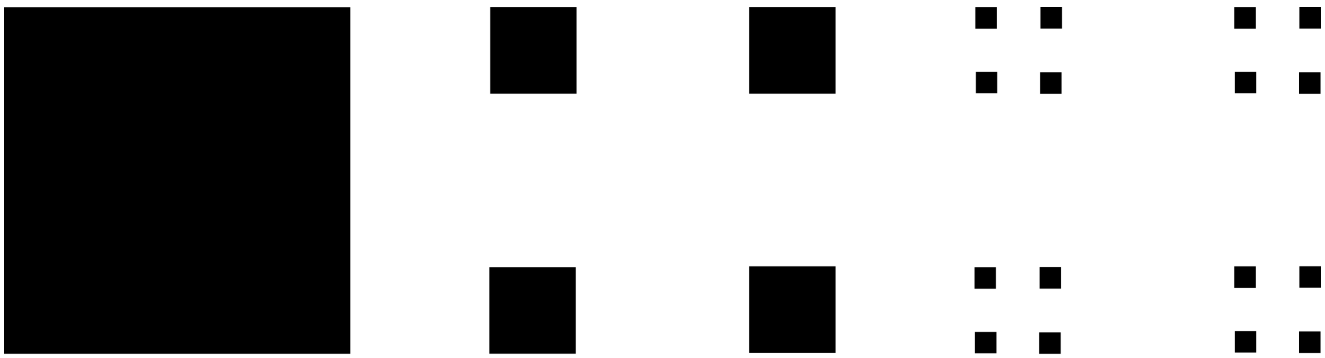
$$E \subset \left(\bigcup_{i=1}^{\infty} \widetilde{\Sigma}_i \right) \cup \widetilde{\Sigma}_0$$

  with $\mathcal{H}^n(\widetilde{\Sigma}_0) = 0$.

- In 1998 B. White showed using a "slicing" argument that Federer's result can be deduced from Besicovitch.

Purely unrectifiable sets

- $Z \subset \mathbb{R}^m$ is n-purely unrectifiable if $0 < \mathcal{H}^n(Z) < \infty$ and $\mathcal{H}^n(\pi_L(Z)) = 0$ for almost every n-dimensional plane $L \subset \mathbb{R}^m$.
- Example: 4-corner Cantor set



  - There exists $C > 1$ such that for each $x \in E_\infty$ and $r \in (0, \sqrt{2})$

$$C^{-1} r \leq \mathcal{H}^1(E_\infty \cap B(x, r)) \leq Cr$$

  - For almost every line L in $\mathbb{R}^2$, $\mathcal{H}^1(\pi_L(E_\infty)) = 0$.

Structure Theorem

Besicovitch (1929-1939) Suppose $X \subset \mathbb{R}^2$ and $0 < \mathcal{H}^1(X) < \infty$. Then

$$X = Y \cup Z$$

- Y is 1-rectifiable.
- Z is 1-purely unrectifiable.

Federer (1947) Suppose $X \subset \mathbb{R}^m$, $n \in \mathbb{N}$ and $0 < \mathcal{H}^n(X) < \infty$. Then

$$X = Y \cup Z$$

- Y is n-rectifiable.
- Z is n-purely unrectifiable.

Rectifiable measures

- A measure μ in $\mathbb{R}^m$ is Radon if it is Borel regular and $\mu(K) < \infty$ for each compact subset $K \subset \mathbb{R}^m$.

- **Federer** A Radon measure μ in $\mathbb{R}^m$ is n-rectifiable iff there exists a family $\{\Sigma_i\}_i$ of Lipschitz images of $\mathbb{R}^n$ such that

$$\mu(\mathbb{R}^m \setminus \bigcup_{i=1}^{\infty} \Sigma_i) = 0$$

- **Mattila & Preiss** A Radon measure μ in $\mathbb{R}^m$ is n-rectifiable iff $\mu \ll \mathcal{H}^n$ and there exists a family $\{\Sigma_i\}_i$ of Lipschitz images of $\mathbb{R}^n$ such that

$$\mu(\mathbb{R}^m \setminus \bigcup_{i=1}^{\infty} \Sigma_i) = 0$$

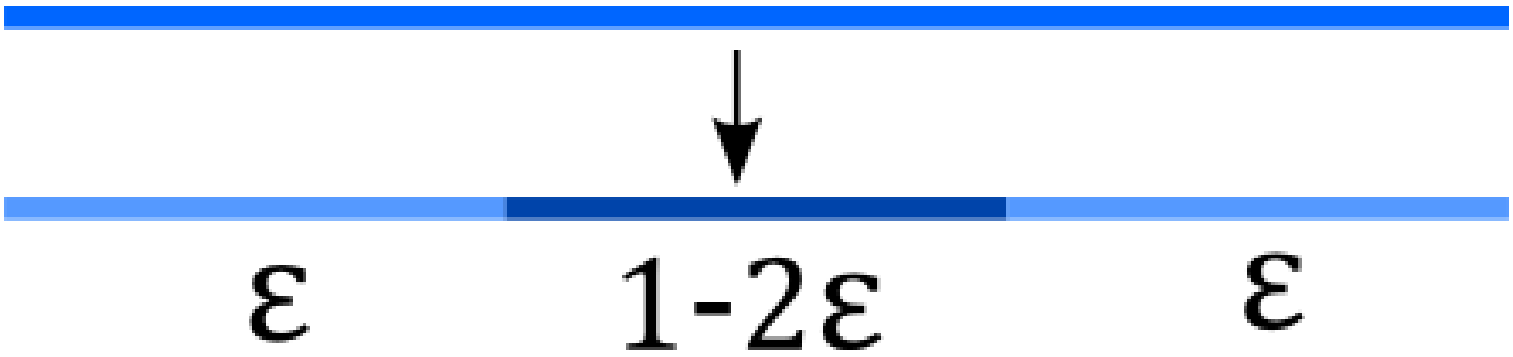
Remarks

- If $E \subset \mathbb{R}^m$ Borel with $0 < \mathcal{H}^n(E) < \infty$, $\mathcal{H}^n \llcorner E$ defined by $\mathcal{H}^n \llcorner E(A) = \mathcal{H}^n(E \cap A)$ is a Radon measure. The rectifiablity of E is equivalent to the rectifiability of $\mathcal{H}^n \llcorner E$ (in the sense of Mattila & Preiss)

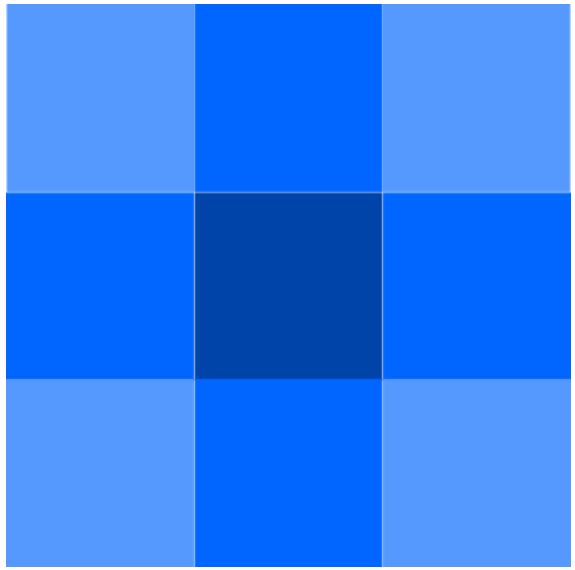
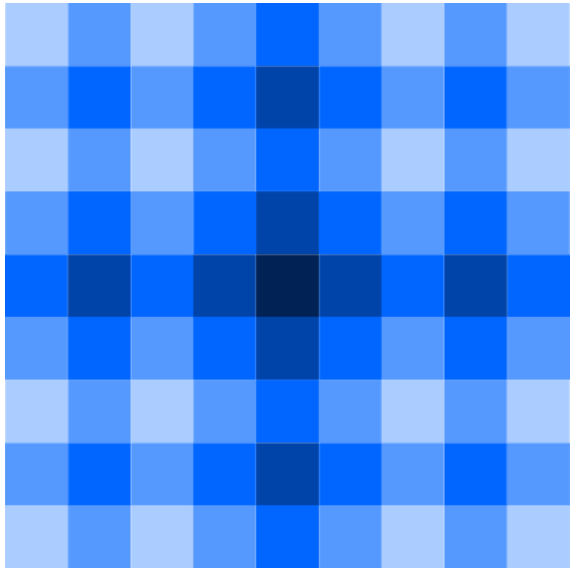
- How different are these two notions of rectifiability?

  - **Garnett-Killip-Schul:** There exists a 1-rectifiable doubling measure μ (in the sense of Federer) supported in all of $\mathbb{R}^m$. Hence $\mathcal{H}^1 \perp \mu$.

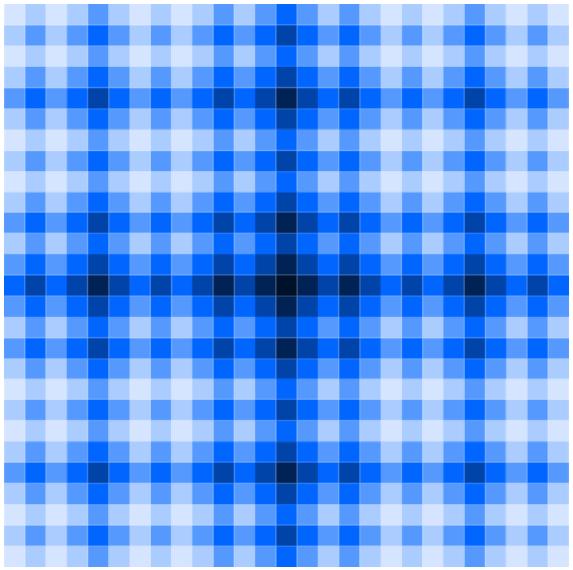
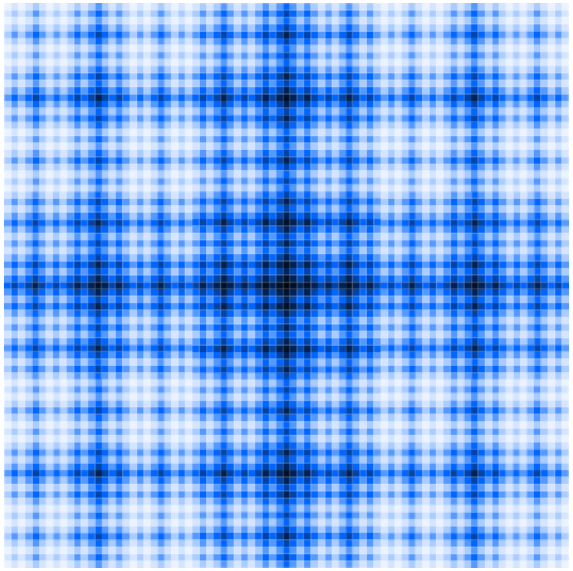
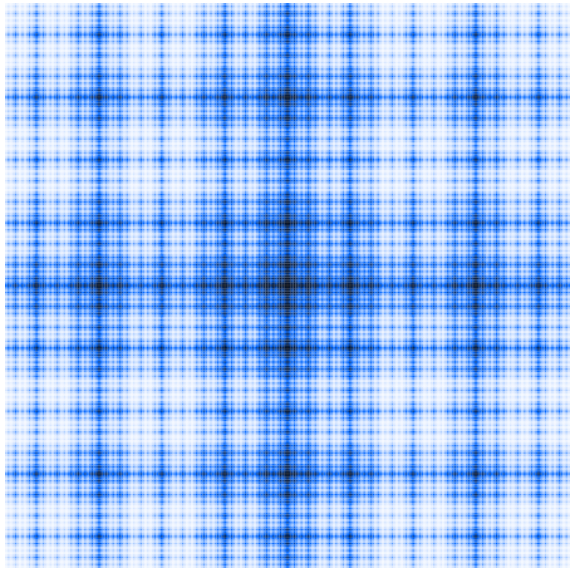
Garnett-Killip-Schul example in $\mathbb{R}^2$



First iterations

First iterations

Courtesy of M. Badger

Properties of the Garnett-Killip-Schul example

- $\operatorname{spt}\mu = \{x \in \mathbb{R}^m : \mu(B(x,s)) > 0,\ \forall s > 0\} = \mathbb{R}^m$
- μ is doubling, i.e. there exists $C > 0$ such that for $x \in \operatorname{spt}\mu = \mathbb{R}^m$ and $s > 0$

$$\mu(B(x,2s)) \leq C\mu(B(x,s))$$

- If $f : \mathbb{R} \to \mathbb{R}^m$ and $\Gamma = \{(x, f(x)) : x \in \mathbb{R}\}$ then $\mu(\Gamma) = 0$.
- There exists a family $\{\Sigma_i\}_i$ of Lipschitz images of $\mathbb{R}$ such that

$$\mu(\mathbb{R}^m \setminus \bigcup_{i=1}^{\infty} \Sigma_i) = 0.$$

- What is the difference between a Lipschitz graph and a Lipschitz image?

Besicovitch question

Let $E \subset \mathbb{R}^2$ such that $0 < \mathcal{H}^1(E) < \infty$. Suppose that

$$\lim_{r \to 0} \frac{\mathcal{H}^1(E \cap B(x,r))}{2r} = 1 \quad \mathcal{H}^1 - a.e.\ x \in E. \tag{1}$$

What can be said about the structure of E?

Examples of sets satisfying (1):

- $E \subset \mathbb{R} \times \{0\}$ Borel, $0 < \mathcal{H}^1(E) < \infty$
- E C^1 curve.

Examples of sets **not** satisfying (1):

- 4-corner Cantor set E_∞.

Densities

- For a Radon measure μ in $\mathbb{R}^m$ we say that the n-density at x exists if

$$\lim_{r \to 0} \frac{\mu(B(x,r))}{r^n} = \theta^n(\mu, x) \in (0, \infty)$$

- What can be said about a Radon measure μ for which the n-density exists μ-a.e. $x \in \mathbb{R}^m$?

Radon measures with n-density in $\mathbb{R}^m$ are n-rectifiable

$\mu = \mathcal{H}^n \llcorner E$	μ
1928-1938 Besicovitch $n = 1,\ m = 2$	
	1944 Morse & Randolph $n = 1,\ m = 2$
E n-rectifiable	μ n-rectifiable

1947 Federer: If $\mathcal{H}^n \llcorner E$ Radon and E n-rectifiable then $\theta^n(\mathcal{H}^n \llcorner E, x) = \omega_n,\ \mathcal{H}^n - a.e.\ x \in E$

ω_n = measure of the unit ball in $\mathbb{R}^n$

Radon measures with n-density in $\mathbb{R}^m$ are n-rectifiable

$\mu = \mathcal{H}^n \llcorner E$	μ
1928-1938 Besicovitch $n = 1,\ m = 2$	
	1944 Morse & Randolph $n = 1,\ m = 2$
	1950 Moore $n = 1,\ m \geq 3$
1961 Marstrand $n = 2,\ m = 3$	
1975 Mattila $n \leq m,\ n, m \in \mathbb{N}$	
	1987 Preiss $n, m \in \mathbb{N}$
E n-rectifiable	μ n-rectifiable

Theorem (Besicovitch) Let $E \subset \mathbb{R}^2$ such that $0 < \mathcal{H}^1(E) < \infty$ and

$$\lim_{r \to 0} \frac{\mathcal{H}^1(E \cap B(x,r))}{2r} = 1 \quad \mathcal{H}^1 - a.e.\ x \in E,$$

then E is 1-rectifiable.

Theorem (Preiss) Let μ be a Radon measure in $\mathbb{R}^m$ such that

$$\lim_{r \to 0} \frac{\mu(B(x,r))}{r^n} = \theta^n(\mu, x) \in (0, \infty)$$

for μ-a.e. $x \in \mathbb{R}^m$. Then μ is n-rectifiable.

Questions

1. Are there conditions that ensure the Federer-rectifiability of a measure? (Badger-Schul $n = 1$).
2. Are there conditions that ensure the Mattila-Preiss-rectifiability of a measure?
3. What happens when we consider the s-density for $s \notin \mathbb{N}$?
4. Does the density ratio $\frac{\mu(B(x,r))}{r^n}$ encode further regularity for μ?

Questions about infinitesimal structure of a measure.
How do we study infinitesimal structure of mathematical objects?

Weak convergence of Radon measures

Let μ, μ_i $(i = 1, 2, \cdots)$ be Radon measures in $\mathbb{R}^m$. The following are equivalent

- $\lim_{i\to\infty} \int \varphi \, d\mu_i = \int \varphi \, d\mu$ for all $\varphi \in C_c(\mathbb{R}^m)$
- $\limsup_{i\to\infty} \mu_i(K) \leq \mu(K)$ for each compact set $K \subset \mathbb{R}^m$ and $\liminf_{i\to\infty} \mu_i(U) \leq \mu(U)$ for each open set $U \subset \mathbb{R}^m$.
- $\lim_{i\to\infty} \mu_i(B) = \mu(B)$ for each Borel set $B \subset \mathbb{R}^m$ with $\mu(\partial B) = 0$.

Notation: $\mu_i \rightharpoonup \mu$.

Compactness of Radon measures

- Let $\{\mu_i\}_i$ be a sequence of Radon measures in $\mathbb{R}^m$ such that for each compact set $K \subset\subset \mathbb{R}^m$

$$\sup_i \mu_i(K) < \infty.$$

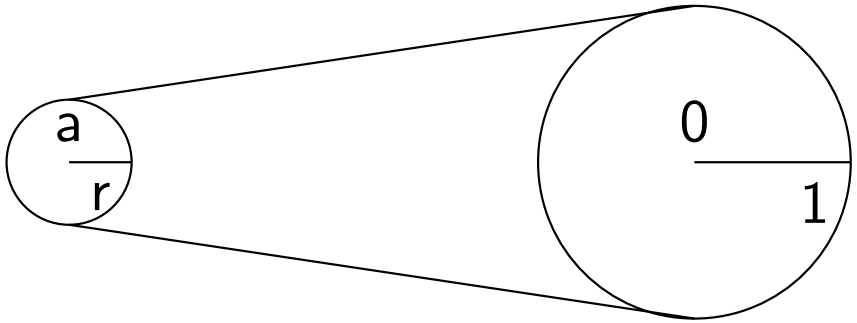
There exists a subsequence $\{\mu_{i_k}\}_{i_k}$ and a Radon measure μ such that

$$\mu_{i_k} \rightharpoonup \mu.$$

- **Remark:** The set of Radon measures in $\mathbb{R}^m$ can be metrized in such a way that it becomes a complete metric space and the metric convergence coincides with the weak convergence.

Tools to study infinitesimal structures: dilations

- The map $T_{a,r}(x) = \frac{x-a}{r}$ maps $B(a,r)$ onto $B(0,1)$.



- The image of a Radon measure μ under $T_{a,r}$ is the Radon measure $T_{a,r}[\mu]$ defined by

$$T_{a,r}[\mu](E) = \mu(rE + a)$$

where $rE + a = \{rx + a : x \in E\}$. Note

$$T_{a,r}[\mu](B(0,1)) = \mu(B(a,r)).$$

Tangent measures

A non-zero Radon measure ν is a tangent measure to μ at $a \in \mathbb{R}^m$ if there exist sequences $\{r_i\}_i$, $r_i \to 0$ and $\{c_i\}_i$, $c_i > 0$ such that

$$c_i T_{a,r_i}[\mu] \rightharpoonup \nu \text{ as } i \to \infty,$$

i.e. $\forall \varphi \in C_c(\mathbb{R}^m)$

$$\lim_{i\to\infty} c_i \int \varphi(\frac{x-a}{r_i})\, d\mu(x) = \int \varphi\, d\nu.$$

The set of all tangent measures to μ at a is denoted by

$$\text{Tan}(\mu, a).$$

Examples

- If $A \subset \mathbb{R}^m$ Borel measurable, $\mu = \mathcal{H}^m \llcorner A$ by Lebesgue density theorem for μ a.e. $a \in \mathbb{R}^m$

$$\text{Tan}(\mu, a) = \{c\mathcal{H}^m : c > 0\}.$$

- If $f \in L^1_{loc}(\mathbb{R}^m)$, $f > 0$ and $\mu(E) = \int_E f(x)\, dx$ then for μ a.e. $a \in \mathbb{R}^m$

$$\text{Tan}(\mu, a) = \{c\mathcal{H}^m : c > 0\}.$$

For $\varphi \in C_c(\mathbb{R}^m)$

$$\begin{aligned} c_i \int \varphi(\frac{x-a}{r_i}) f(x)\, dx &= c_i \int \varphi(y) f(r_i y + a) r_i^m\, dy \\ &= c_i r_i^m \int \varphi(y)(f(r_i y + a) - f(a))\, dy \\ &\quad + c_i r_i^m f(a) \int \varphi(y)\, dy \\ &\dots \end{aligned}$$

Examples

- If $A \subset \mathbb{R}^m$ Borel measurable, $\mu = \mathcal{H}^m \llcorner A$ by Lebesgue density theorem for μ a.e. $a \in \mathbb{R}^m$

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- If $f \in L^1_{loc}(\mathbb{R}^m)$, $f > 0$ and $\mu(E) = \int_E f(x)\, dx$ then for μ a.e. $a \in \mathbb{R}^m$

$$\text{Tan}(\mu, a) = \{c\mathcal{H}^m : c > 0\}.$$

- If $\Sigma \subset \mathbb{R}^m$ a C^1 n-dimensional submanifold and $\mu = \mathcal{H}^m \llcorner \Sigma$ then

$$\text{Tan}(\mu, a) = \{c\mathcal{H}^n \llcorner (T_a\Sigma - a) : c > 0\}$$

  where $T_a\Sigma$ is the tangent plane to Σ at a.

- Mattila constructed a family of examples where the tangents are not unique and none of them are flat either.

Existence

- Preiss (1987) For any Radon measure μ, $\mathrm{Tan}(\mu, a) \neq \emptyset$, μ- a.e. $a \in \mathbb{R}^m$.
- Let μ be a Radon measure in $\mathbb{R}^m$, if for $a \in \mathbb{R}^m$

$$c_a = \limsup_{r \to 0} \frac{\mu(B(a, 2r))}{\mu(B(a, r))} < \infty \quad (\mu \text{ asymptotically doubling at } a)$$

then every sequence $\{r_i\}_i$, $r_i \to 0$ contains a subsequence $\{r_{i_k}\}_{i_k}$ such that

$$\mu_{a, r_{i_k}} = \frac{1}{\mu(B(a, r_{i_k}))} T_{a, r_{i_k}}[\mu] \rightharpoonup \nu \neq 0 \quad \text{and} \quad \nu \in \mathrm{Tan}(\mu, a).$$

Moreover any tangent measure is obtained this way up to a constant.