Math 524

Homework due 12/03/14

Reading from Folland: Chapter 3: §4 and §5.

Exercises from Folland:

Chapter 3: Section 4, problem 22 (use Mf as defined in class rather than Hf to do this problem).

Problem 1: Prove that if a measurable subset $E \subset [0,1]$ satisfies $m(E \cap I) \geq \alpha m(I)$ for some $\alpha > 0$ and all intervals I in [0,1], then E has measure 1.

Problem 2: If $a,\ b > 0$, let

$$f(x) = x^a \sin(x^{-b})\chi_{(0,1]}(x).$$

Prove that $f \in BV[0,1]$ if and only if $a > b$. Then by taking $a = b$, construct (for each $0 < \alpha < 1$) a Hölder continuous function with exponent α that is a function satisfying

$$|f(x) - f(y)| \leq C|x - y|^\alpha \text{ for } x, y \in [0,1]$$

but which is not of bounded variation.

Hint: Note that if $h > 0$, the difference $|f(x+h) - f(x)|$ can be estimated by $C(x+h)^\alpha$ or $C'h/x$ by the mean value theorem. The consider two cases, whether $x^{a+1} \geq h$ or $x^{a+1} < h$. What is the relationship between α and a?

Problem 3: Suppose μ is σ-finite and $f_n \to f$ μ-a.e.. Show that there exist measurable sets $\{E_j\}$ in X such that $\mu\left((\bigcup_{j=1}^\infty E_j)^c\right) = 0$ and $f_n \to f$ uniformly on each E_j.

Problem 4: Let (X, d) be a metric space. Let μ be a doubling measure on $\mathcal{B}_X$, i.e. there exists a constant $C \geq 1$ such that $\forall x \in X$ and $r > 0$

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

Assume that $\mu(B(x_0, 1)) < \infty$ for some $x_0 \in X$.

1. Show that if K is compact then $\mu(K) < \infty$.
2. Assume $X = \mathbb{R}^n$ and d is the Euclidean distance. Prove that if $E \in \mathcal{B}_{\mathbb{R}^n}$ with $\mu(E) < \infty$ then given $\epsilon > 0$ there exists a compact set K with $K \subset E$ and $\mu(E \backslash K) < \epsilon$.

   Note: μ is not necessarily Lebesgue measure.