Math 524

Homework due 11/26/14

Reading from Folland: Chapter 2: §5, §6 and §7.

Exercises from Folland:

Chapter 2: Section 5, problems 46 and 48.

Problem 1: Suppose F is a closed set in $\mathbb{R}$, whose complement has finite measure, and let $\delta(x)$ denote the distance from x to F, that is

$$\delta(x) = d(x, F) = \inf\{|x - y| : \ y \in F\}.$$

Consider

$$I(x) = \int_{\mathbb{R}} \frac{\delta(y)}{|x - y|^2}\, dy.$$

1. Prove that δ is continuous, by showing that it satisfies the Lipschitz condition
$$|\delta(x) - \delta(y)| \leq |x - y|.$$
2. Show that $I(x) = \infty$ for each $x \notin F$.
3. Show that $I(x) < \infty$ for $a.\,e.\ x \in F$. This may be surprising in view of the fact that the Lipschitz condition cancels only one power of $|x - y|$ in the integrand of I. [Hint: Investigate $\int_F I(x)\, dx$.]

Problem 2: (Tchebychev Inequality) Suppose that $f : \mathbb{R}^n \to [0, \infty]$ and $f \in L^1(\mathbb{R}^n)$. If $\alpha > 0$ and $E_\alpha = \{x : f(x) > \alpha\}$, prove that

$$m(E_\alpha) \leq \frac{1}{\alpha} \int f\, dm$$

Problem 3: Suppose that $f : \mathbb{R}^n \to [0, \infty]$. Let

$$E_{2^k} = \{x : \ f(x) > 2^k\} \text{ and } F_k = \{x : 2^k < f(x) \leq 2^{k+1}\}.$$

If f is finite almost everywhere, then

$$\bigcup_{k=-\infty}^{\infty} F_k = \{x : \ f(x) > 0\},$$

and the F_k are disjoint. Prove that the following are equivalent

1. $f \in L^1(\mathbb{R}^n)$
2. $\displaystyle\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$
3. $\displaystyle\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty$

Use this result to verify the following assertions.

1. If $f(x) = |x|^{-a}\chi_{\{|x|\leq 1\}}$, then $f \in L^1(\mathbb{R}^n)$ if and only if $a < n$.
2. If $g(x) = |x|^{-b}\chi_{\{|x|>1\}}$, then $g \in L^1(\mathbb{R}^n)$ if and only if $b > n$.

Problem 4: Suppose that $f \in L^1(\mathbb{R}^n)$. For each $\alpha > 0$ let $E_\alpha = \{x : |f(x)| > \alpha\}$. Prove that

$$\int_{\mathbb{R}^n} |f(x)|\, dx = \int_0^\infty m(E_\alpha)\, d\alpha.$$