Math 524

Homework due 10/08/14

Reading from Folland: Chapter 1, sections 1, 3, 4.

Problem 1. Properties of the exterior measure in $\mathbb{R}^d$.

1. Prove that the exterior measure of closed cube in $\mathbb{R}^d$ is its volume.
2. Prove that if $E \subset \mathbb{R}^d$ is a countable union of closed cubes such that their interiors are disjoint, that is
$$E = \cup_{i=1}^{\infty} Q_i \text{ with } \operatorname{int} Q_i \cap \operatorname{int} Q_j = \emptyset$$
then
$$m_*(E) = \sum_{j=1}^{\infty} m_*(Q_j).$$

Problem 2. Prove that if $E \subset \mathbb{R}^d$ and $E = E_1 \cup E_2$, with $d(E_1, E_2) > 0$ then
$$m_*(E) = m_*(E_1) + m_*(E_2).$$

Problem 3. Let F be a subset of $[0, 1]$ constructed in the same manner as the Cantor ternary set except that each of the intervals removed at the nth step has length $\alpha 3^{-n}$ with $0 < \alpha < 1$. Then F is closed, F^c is dense in $[0, 1]$ and $m_* F = 1 - \alpha$. Such set F is called a *generalized Cantor set.*

Problem 4. Let (X, ρ) be a non-empty complete metric space. Let $f : X \to X$ be such that there exists $\lambda \in (0, 1)$ satisfying
$$\rho(f(x), f(y)) \leq \lambda \rho(x, y) \text{ for all } x, y \in X.$$
Prove that there exists a unique point $u \in X$ such that $f(u) = u$. (Hint: let $x \in X$, and consider the sequence $x,\ f(x),\ f(f(x)), ...$). This result is known as Banach's fixed-point theorem. It implies several existence theorems in the theory of partial differential equations.

Problem 5 (Prelim 2001) Let (X, d) be a metric space and $K \subset X$ be compact. Let $\{G_i\}_{i \in I}$ be an open cover of K. Show that there exits $\epsilon > 0$ such that every subset S of K of diameter $< \epsilon$ is contained in one of the sets G_i (i.e. that there exists $i = i(S)$ such that $S \subset G_i$).

Problem 6^***.** Show that any open set $\Omega \subset \mathbb{R}^d$ can be written as the union of dyadic cubes, so that $\Omega = \cup Q_j$, with the following properties.

1. The $Q_j's$ have disjoint interiors.
2. $d(Q_j, \Omega^c) \sim \ell(Q_j)$, where $\ell(Q_j)$ is the side length of Q_j. This means that there are positive constants c and C so that

$$c\ell(Q_j) \leq d(Q_j, \Omega^c) \leq C\ell(Q_j).$$