1. Geometric Measure Theory, Professor Tatiana Toro.

Lecture 1: March 31, 2014

Consider $E \subset \mathbb{R}$ with $0 < m(E) < \infty$, where m is Lebesgue measure. Then by the differentiation theorem, we have for m-a.e. $x \in E$,

$$\lim_{r \to 0} \frac{m((x-r, x+r) \cap E)}{2r} = 1$$

Besicovitch studied the following situations: When $E \subset \mathbb{R}^2$, and $\mathcal{H}^1 \llcorner E(A) := \mathcal{H}^1(E \cap A)$, (where $\mathcal{H}^1$ is one-dimensional Hausdorff measure), what can one say about E if for $\mathcal{H}^1$-a.e. $x \in E$ we have

$$\lim_{r \to 0} \frac{\mathcal{H}^1(E \cap B(x,r))}{2r} = 1$$

From 1928-1938, it was found that if the above holds, and $0 < \mathcal{H}^1(E) < \infty$, then $E \subset \bigcup_{i=1}^{\infty} f_i(\mathbb{R}) \cup E_0$, where $\mathcal{H}^1(E_0) = 0$ and the f_i are Lipschitz.

Recall $2^X = \mathcal{P}(X)$.

Definition 1. *A mapping $\mu : 2^X \to [0, \infty]$ is a measure, (outer), on X if: i) $\mu(\varnothing) = 0$, ii) $\mu(A) \leq \sum_1^\infty \mu(A_i)$ when $A \subset \cup_1^\infty A_i$.*

Remark 2. *If $A \subset B$ then $\mu(A) \leq \mu(B)$.*

Definition 3. *$A \subset X$ is μ-measurable if $\mu(B) = \mu(B \cap A) + \mu(B \cap A^c)$ for all $B \subset X$.*

Remark 4. *If A is μ-measurable, the A^c is also.*

Theorem 5. *Let $\{A_k\}_1^\infty$ be μ measurable, then:*

1. *$\cup_1^\infty A_i$ and $\cap_1^\infty A_i$ are measurable.*
2. *$\mu(\cup A_i) = \sum \mu(A_i)$ if the A_i are pairwise disjoint.*
3. *If $A_k \subset A_{k+1}$ then $\mu(\cup A_i) = \lim \mu(A_k)$.*
4. *If $A_{k+1} \subset A_k$ and $\mu(A_1) < \infty$ then $\mu(\cap A_k) = \lim \mu(A_k)$.*

Recall that $\mathcal{A}$ is a σ-algebra over X if $\mathcal{A} \subset 2^X$, both X and $\varnothing$ are in $\mathcal{A}$, and $\mathcal{A}$ is closed under complements and countable unions. Whence the μ-measurable sets form a σ-algebra.

Definition 6. *Let (X, d) be a metric space. The* Borel σ-algebra *generated by the open sets of X is the smallest σ-algebra containing the open sets.*

Definition 7.

1. *A(n) (outer) measure μ on X is* regular *if for each $A \subset X$ there is a μ-measurable set $B \supset A$ with $\mu(A) = \mu(B)$.*
2. *A measure on (X, d) is called a* Borel measure *if all Borel sets are measurable.*
3. *A measure on (X, d) is* Borel regular *if it is Borel and for every $A \subset X$ there is a Borel set $B \supset A$ with $\mu(A) = \mu(B)$.*
4. *A measure μ on (X, d) is* Radon *if it is Borel regular and every compact set has finite measure.*

Theorem 8. *Let μ be a regular measure on X, then if $\{A_k\}_1^\infty \subset 2^X$ and $A_k \subset A_{k+1}$, then $\mu(\cup A_k) = \lim \mu(A_k)$.*

Remark 9. *Note that the A_k need not be measurable in this case.*

Theorem 10. *Let μ be a Borel regular measure on (X, d) and A a μ-measurable set with finite measure. Then $\mu \llcorner A$ is Radon.*

Theorem 11. *Let μ be Borel regular on (X, d). Suppose $X = \cup_1^\infty V_i, V_i$ open with finite measure. Then:*

- $\mu(A) = \inf_{A \subset U,\ \mathrm{U\ open}} \mu(U)$, *for each* $A \subset X$.
- $\mu(A) = \sup_{C \subset A,\ \mathrm{C\ closed}} \mu(C)$ *for* A μ*-measurable.*

For example, d-dimensional Lebesgue measure.

Remark 12. *: Assume A is μ measurable with finite measure and (1) holds in Theorem 11. Then for any $\epsilon > 0$ there is an open set $U \supset A$ with $\mu(U) < \mu(A) + \epsilon$. Thus $\mu(U) = \mu(U \cap A) + \mu(U \cap A^c) = \mu(A) + \mu(U \setminus A) < \mu(A) + \epsilon$, so $\mu(U \setminus A) < \epsilon$.*

We show that the statement (1) in the theorem implies (2).

1) $\implies$ 2). If A is μ-measurable, then so is A^c, and by following the remark, we have that for a given $\epsilon > 0$, we have an open $U \supset A^c$ with $\mu(U \setminus A^c) = \mu(U \cap A) < \epsilon$. Then we have

$$\mu(A) = \mu(A \cap U^c) + \mu(A \setminus U^c) = \mu(U^c) + \mu(A \setminus U^c)$$

which shows that $\mu(U^c) \leq \mu(A) \leq \mu(U^c) + \epsilon$. □

proof of (1) when X has finite measure. Assume $\mu(X) < \infty$ finite and let A be μ-measurable (Borel). Consider $\mathfrak{M} = \{A \text{ Borel : (1) holds}\}$. Then $X, \varnothing \in \mathfrak{M}$, and $\mathfrak{M}$ is closed under countable unions and intersections. For any $\epsilon > 0$, and for any k there is an open $U_k \supset A_k$ with $\mu(U_k \setminus A_k) < \epsilon$. Observe that we have the containment monotonicity property $\cup_1^{j+1} U_k \subset \cup_1^j U_k$, and the same with $\cap_1^j A_k$. Then $\mu(\cap_1^\infty A_k) = \lim_{j\to\infty} \mu(\cap_{k=1}^j A_k)$, so that $\mu(\cap_{k=1}^j U_k) \leq \epsilon + \mu(\cap A_k)$ for large j. Consequently, we have that

$$\mu(\cap_{k=1}^\infty A_k) \leq \mu(\cap_{k=1}^j U_k) \leq 2\epsilon + \mu(\cap_{k=1}^\infty A_k)$$

Thus (1) is closed under countable intersections. Notice that the open sets are in $\mathfrak{M}$, and all the closed sets are too: Consider $\mathfrak{A} = \{A \in \mathfrak{M} : X \setminus A \in \mathfrak{M}\}$, and observe that any closed set is the countable intersection of open sets. If C is closed, it can be written as the intersection of $U_i = \cup_{x \in C} B(x, \frac{1}{2^i})$. Since the U_i are in $\mathfrak{M}$, so is C.

To finish off the proof of the theorem when X may not have finite measure, apply the previous result to $\mu \llcorner V_j$. □

Corollary 13. *Let μ be a Radon measure on (X, d), a separable locally compact space, then for A μ-measurable, we have that (1) holds and*

$$\mu(A) = \sup_{K \subset A,\ \mathrm{K\ compact}} \mu(K).$$

Recall Caratheodory's Theorem, which says that if μ is a measure on (X, d) such that for all $A, B \subset X$, $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $\mathrm{d}(A, B) > 0$, then μ is Borel regular.

2. Lecture 2. April 2nd, 2014

We now start discussing covering lemmas, beginning with the following

Theorem 14 (Vitali)**.** *Let $\mathcal{F}$ be a family of nondegenerate closed balls such that*

$$\sup\{\operatorname{diam} \mathrm{B} : \mathrm{B} \in \mathcal{F}\} = \mathrm{R} < \infty.$$

Then there exists a pairwise disjoint subcollection $\mathcal{G} \subset \mathcal{F}$ such that

$$\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{B\in\mathcal{G}} \widehat{B},$$

along with the property that for every $B \in \mathcal{F}$ there is an $S \in \mathcal{G}$ with $B \cap S \neq \varnothing$, with $B \subset \widehat{S}$. Here $\widehat{B}$ is a ball with five times the radius of B with the same center.

Proof. Let $\mathcal{F}_j = \left\{B \in \mathcal{F} : 2^{-j}R < \operatorname{diam} \mathrm{B} \leq 2^{-\mathrm{j}+1}\mathrm{R}\right\}$. Now, set $\mathcal{G}_1$ as a maximal disjoint subcollection of $\mathcal{F}_1$. Given $\mathcal{G}_1, \ldots, \mathcal{G}_{j-1}$ set $\mathcal{G}_j$ as the maximal disjoint subcollection of $\left\{B \in \mathcal{F}_j : B \cap B' = \varnothing \, \forall B' \subset \bigcup_{i=1}^{j-1} \mathcal{G}_i\right\}$. We show that letting $\mathcal{G} = \cup\mathcal{G}_i$ has the desired properties. Clearly, we have $\mathcal{F} = \cup\mathcal{F}_i$, and if a particular B is in $\mathcal{F}_j$, there is a $D \in \cup_1^j \mathcal{G}_i$ with $D \cap B \neq \varnothing$. Since $\operatorname{diam} \mathrm{B} \leq 2^{-\mathrm{j}+1}\mathrm{R} \leq 2 \operatorname{diam} \mathrm{D}$, and as $\operatorname{diam} \mathrm{D} \geq 2^{-\mathrm{j}}\mathrm{R}$, we have $B \subset \widehat{D}$. Hence we also have $\bigcup_{B\in\mathcal{F}_k} B \subset \bigcup_{B\in\mathcal{G}} \widehat{B}$, and the result follows. □

Definition 15. *A collection $\mathcal{F}$ of (closed) nondegenerate balls* cover *A if $A \subset \cup_{B\in\mathcal{F}} B$. We say that $\mathcal{F}$ is a* ***fine cover*** *of A if $\mathcal{F}$ covers A, and for each x in A, we have*

$$\inf_{B\in\mathcal{F}} \{\operatorname{diam}(B) : x \in B\} = 0.$$

Corollary 16. *Let $\mathcal{F}$ be the collection of nondegenerate closed balls. Let $\mathcal{F}$ be a fine cover of A, and assume that*

$$\sup\{\operatorname{diam}(B) : B \in \mathcal{F}\} = R < \infty,$$

then there is a pairwise disjoint subcollection of $\mathcal{F}$, call it $\mathcal{G}$, such that for any natural number N,

$$A \setminus \bigcup_1^N B_i \subset \bigcup_{B\in\mathcal{G}\setminus\{B_i\}_1^N} \widehat{B}.$$

Remark 17. *Note that the assumption above that R be finite is actually redundant. In fact since the cover is fine one could replace the collection $\mathcal{F}$ with an appropriate subcollection whose balls have uniformly bounded diameter.*

Proof of Corollary. Let $x \in A \setminus \cup_1^N B_i$, where the B_i are in $\mathcal{G}$. Then there is a $B \in \mathcal{F}$ such that x is in B and, by fineness of the cover, we can choose B so $B \cap B_i = \varnothing, i = 1, \ldots, N$. By Vitali, then, there is a D in the collection $\mathcal{G}$, distinct from the B_i, with $B \subset \widehat{D}$, and $B \cap D \neq \varnothing$. □

Theorem 18 (Besicovitch's Covering Theorem)**.** *There is a constant N_n, depending only on n, with the property that if $\mathcal{F}$ is any collection of nondegenerate closed balls in $\mathbb{R}^n$ with diameters uniformly bounded by a constant $D < \infty$, i.e.*

$$\sup\{\operatorname{diam}(B) : B \in \mathcal{F}\} = D < \infty,$$

and A is the collection of the centers of the balls, then there exists $\mathcal{G}_1, \ldots, \mathcal{G}_{N_n} \subset \mathcal{F}$ with each $\mathcal{G}_i$ a ***countable*** *collection of pairwise disjoint balls and*

$$A \subset \bigcup_1^{N_n} \bigcup_{B \in \mathcal{G}_i} B.$$

Remark 19. *Although Besicovitch's covering theorem may seem to be less useful than Vitali's covering theorem, there are in fact different uses, (and required hypthoses!), of these two theorems. Vitali's theorem can be used in an arbitrary metric space, and it gives one disjoint family of balls whose dilations cover the original set but are no longer disjoint. It is used in case we are dealing with a doubling measure. Besicovitch's theorem can be used in $\mathbb{R}^n$, and gives us disjoint families with finite overlap of at most N_n. It is used in case we are dealing with a Radon measure.*

Sketch of the proof of Besicovitch. If A is bounded, choose $B_1 = B(a_1, r_1) \in \mathcal{F}, r_1 \geq \frac{3}{4}\frac{D}{2}$, and inductively choose $B_j, j \geq 2$ as follows: Set$A_j = A \setminus \cup_1^{j-1} B_j$. If A_j is empty, we stop at $J = j - 1$, and $A \subset \cup_1^J B_i$. If not, find $a_j \in A_j$ such that

$$B_j = B(a_j, r_j), r_j \geq \frac{3}{4}\frac{1}{2} \sup \{r : B(a, r) \in \mathcal{F},\, a \in A_j\}.$$

If A_j is not empty for all j, then $J = +\infty$.

- Claim 1: If $j > i$, then $r_j \leq \frac{4}{3} r_i$.
- Claim 2: $\left\{B(a_j, \frac{r_j}{3})\right\}_{i=1}^J$ is a disjoint collection.
- Claim 3: If J is $+\infty$, then $r_j \to 0$.
- Claim 4: If $A \subset \bigcup_{j=1}^J B(a_j, r_j)$.

Fix $k \in \mathbb{N}$, set $I = \{j : 1 \leq J \leq k,\, B_j \cap B_k \neq \varnothing\}$ and $K = I \cap \{j : r_j \leq 3r_k\}$.

- Claim 5: The cardinality of K is not more than 20^n.
- Claim 6: There is an L_n such that $\text{Card}(I \setminus K) \leq L_n$. To show claim 6, one uses geometry and trigonometry of the triangles between centers of balls.

□

Definition 20. *A* doubling measure μ *is a Borel regular measure with a constant $0 < c < \infty$ such that $\mu(B(x, 2r)) \leq c\mu(B(x, r))$ for all x, r, and $\mu(B(x_0, 1)) < \infty$ for some $x_0 \in X$.*

Remark 21. *This last condition guarantees that bounded sets have bounded measure.*

Theorem 22. *Let (X, d) be a metric space, and let μ be a doubling measure, $\mathcal{F}$ a fine cover of $A \subset X$ by nondegenerate closed balls. Then there is a countable subcollection $\mathcal{G} = \{B_i\}_1^\infty \subset \mathcal{F}$ of disjoint balls such that $\mu(A \setminus \cup_1^\infty B_i) = 0$.*

Proof. Note that the doubling measure property gives $\mu(\widehat{B}) \leq \mu(8B) \leq c^3\mu(B)$. We start with the case when the measure of A has finite, (nonzero) measure. First, notice that there is then an open set $U \supset A$ with $\mu(U) < \mu(A) + \epsilon$, and replacing $\mathcal{F}$ with the collection $\mathcal{F}' := \{B \in \mathcal{F} : B \subset U\}$, we can replace X with U, i.e., we may assume that X has finite measure. By Vitali's corollary there exists a $\mathcal{G} \subset \mathcal{F}$ such that $\{B_i\} \subset \mathcal{G}$ and

$$A \setminus \bigcup_1^N B_i \subset \bigcup_{B \in \mathcal{G} \setminus B_i 1^N} \widehat{B}$$

Since $\mu(X) < \infty$ and $\mathcal{G}$ is a disjoint family, $\mathcal{G}$ is countable, since nondegenerate balls have positive measure when the measure is doubling. Then

$$\mu(A \setminus \bigcup_{1}^{N} B_i) \leq \mu(\bigcup_{N+1}^{\infty} \widehat{B_j}) \leq c^3 \sum_{N+1}^{\infty} \mu(B_j)$$

since

$$\mu(\bigcup_{1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mu(B_i) \leq \mu(U) < \infty,$$

letting $N \to \infty$ gives the result in the case $\mu(X) < \infty$. To extend to the case when X has infinite measure, we write X as a countable union of annuli by taking balls with radius going to infinity. We do so in such a way that the boundary of the annuli has measure zero. That such a sequence of annuli exists follows from the doubling property, which in turn gives that each bounded ball has finite measure. Thus there must never be an uncountable number of spheres with positive measure, (spheres here are boundaries of balls, for simplicity consider them to have the same center). We can now write $X \overset{a.e.}{=} \cup_1^\infty D_i$, where the D_i are the open annuli. Now apply the above case to $\mu \llcorner D_i$. □

The next theorem gives a similar result for Radon measures in $\mathbb{R}^n$, which are not necessarily doubling.

Theorem 23. *Let μ be a Radon measure in $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$. If $\mathcal{F}$ is a fine collection of nondegenerate closed balls with centers in the set A, then there is a countable subcollection $\mathcal{G} = \{B_i\}_1^\infty \subset \mathcal{F}$ with $\mu(A \setminus \cup B_i) = 0$.*

3. Lecture 3: April 7, 2014

Last week we talked about two covering theorems. We finished by talking about the following theorem:

Theorem 24. *Let μ be a Radon measure in $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$. Define*

$$\mathcal{F} = \{B(x,r) \,:\, x \in A\}.$$

Suppose $\inf\{r \,:\, B(x,r) \in \mathcal{F}\} = 0$ *for all $x \in A$ (i.e. $\mathcal{F}$ is a fine cover of A). Then there exists a countable collection of disjoint balls $\mathcal{G} \subset \mathcal{F}$ such that $\mu\left(A \setminus \bigcup_i B_i\right) = 0$ where $\{B_i\}_{i=1}^{\infty} = \mathcal{G}$.*

Remark: In the previous proof, we used a doubling measure, but we do not have this.

Proof. Since μ is a Radon measure and $\mathbb{R}^n$ is locally compact, $\mu(B(0,r)) < \infty$ for all $r > 0$ (Remember balls are considered closed, unless otherwise indicated). Then $\mu(\partial B(0,s)) > 0$ for only countably many s. This means there exists a sequence of r_i increasing to ∞ such that $\mu(\partial B(0,r_i)) = 0$. One sees that in measure

$$A = \bigcup_{i=1}^{\infty} (A \cap \text{int } B(0,r_{i+1})) \setminus B(0,r_i).$$

Essentially, one is breaking the set A into sets which only live in an annulus. This procedure yields a decomposition of A (in measure) by sets which are disjoint and and contained in disjoint open sets. Hence,

$$\mu(A) = \sum_{i=1}^{\infty} \mu((A \cap \text{int } B(0,r_{i+1})) \setminus B(0,r_i)).$$

Note that because $\mu(B(0,r)) < \infty$, $\mu((A \cap \text{int } B(0,r+1)) \setminus B(0,r)) \leq \mu(B(0,r+1)) < \infty$.

Assume that $0 < \mu(A) < \infty$. Recall that in the Besicovitch's covering lemma, there exists a $N = N_n$ that depends only on n (i.e. where n is the dimension). This is called the Besicovitch number. Next, one see that there exists an open U such that $A \subset U$ and $\mu(U) < \mu(A)\left(1 + \frac{1}{4N_n}\right)$. Let

$$\tilde{\mathcal{F}} = \{B \in \mathcal{F} \,:\, B \subset U \text{ and diam } B < 1\}.$$

By the assumption that $\inf\{r \,:\, B(x,r) \in \mathcal{F}\} = 0$ for all $x \in A$, $\tilde{\mathcal{F}}$ is a fine cover of A for $\mathcal{F}$ is a fine cover and $\sup\{\text{diam } B \,:\, B \in \tilde{\mathcal{F}}\} < 1$. Applying Besicovitch's covering lemma, there exists sub-collections $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{N_n} \subset \tilde{\mathcal{F}}$ such that $\mathcal{G}_i$ is a countable collection of pairwise disjoint balls and

$$A \subset \bigcup_{i=1}^{N_n} \Big(\bigcup_{B \in \mathcal{G}_i} B \Big).$$

There exists an $i^* \in \{1, \ldots, N_n\}$ such that

$$\mu\Big(\bigcup_{B \in \mathcal{G}_{i^*}} B \Big) = \max\left\{ \mu\Big(\bigcup_{B \in \mathcal{G}_i} B \Big) \,:\, i \in \{1, \ldots, N_n\} \right\}.$$

Hence,

$$\mu(A) \leq \sum_{i=1}^{N_n} \mu\big(\bigcup_{B\in\mathcal{G}_i} B\big) \leq N_n \mu\big(\bigcup_{B\in\mathcal{G}_{i^*}} B\big).$$

Because every $B \in \mathcal{G}_{i^*}$ is contained in U and the balls in $\mathcal{G}_{i^*}$ are disjoint,

$$\mu\big(\bigcup_{B\in\mathcal{G}_{i^*}} B\big) = \sum_{B\in\mathcal{G}_{i^*}} \mu(B) \leq \mu(U) \leq \big(1+\frac{1}{4N_n}\big)\mu(A). \tag{25}$$

By (25) and $\mu(A) < \infty$, $\sum_{B\in\mathcal{G}_{i^*}} \mu(B) < \infty$; thus there exists a finite subcollection $\mathcal{G}'_{i^*} \subset \mathcal{G}_{i^*}$ such that

$$N_n \mu\left(\bigcup_{B\in\mathcal{G}_{i^*}} B\right) \leq 2N_n \mu\left(\bigcup_{B\in\mathcal{G}'_{i^*}} B\right).$$

It follows that

$$\mu(A) \leq N_n \mu\left(\bigcup_{B\in\mathcal{G}_{i^*}} B\right) \leq 2N_n \mu\left(\bigcup_{B\in\mathcal{G}'_{i^*}} B\right).$$

Define $A_1 = A \setminus \bigcup_{B\in\mathcal{G}'_{i^*}} B$. Since A and all the balls were contained in U,

$$\begin{aligned}\mu(A_1) \leq \mu(U) - \mu\big(\bigcup_{B\in\mathcal{G}'_{i^*}} B\big) &\leq \mu(U) - \frac{1}{2N_n}\mu(A) \\ &\leq \big(1+\frac{1}{4N_n}-\frac{1}{2N_n}\big)\mu(A) = \big(1-\frac{1}{4N_n}\big)\mu(A).\end{aligned}$$

Therefore,

$$\mu(A_1) \leq \big(1-\frac{1}{4N_n}\big)\mu(A).$$

Let $\theta = 1 - \frac{1}{4N_n}$ so that $\mu(A_1) \leq \theta\mu(A)$. Note that $\theta < 1$ and so we have shrunk the set. Moreover, one sees that $A_1 \subset U \setminus \bigcup_{B\in\mathcal{G}'_{i^*}} B$. Because $\mathcal{G}'_{i^*}$ is finite, then the finite union of closed sets is closed; hence, A_1 is contained in the open set $U \setminus \bigcup_{B\in\mathcal{G}'_{i^*}} B$ and so there exists an open $U_1 \supset A_1$ such that $U_1 \subset U \setminus \bigcup_{B\in\mathcal{G}'_{i^*}} B$ and $\mu(U_1) \leq \left(1+\frac{1}{4N_n}\right)\mu(A)$. Set $\tilde{\mathcal{F}}_1 = \{B \in \tilde{\mathcal{F}} : B \subset U_1\}$ and note that $\tilde{\mathcal{F}}_1$ is a fine cover of A_1. Assume that $\mathcal{G}'_{i^*} = \{B_j\}_{j=1}^{M_1}$. By repeating the procedure (i.e. applying Besicovitch and getting a bound on $\mu(A_1)$), one gets $\{B_j\}_{j=M_1+1}^{M_2} \subset \tilde{\mathcal{F}}_1$ such that

$$\mu\big(A_1 \setminus \bigcup_{j=M_1+1}^{M_2} B_j\big) \leq \theta\mu(A_1) \leq \theta^2\mu(A).$$

Because $A \setminus \bigcup_{j=1}^{M_2} B_j = \left(A \setminus \bigcup_{j=1}^{M_1} B_j\right) \cap \left(A_1 \setminus \bigcup_{j=M_1+1}^{M_2} B_j\right)$, then

$$\mu\big(A \setminus \bigcup_{j=1}^{M_2} B_j\big) \leq \mu\big(A_1 \setminus \bigcup_{j=M_1+1}^{M_2} B_j\big) \leq \theta^2\mu(A).$$

Continuing this procedure k times gives the existence of $\{B_j\}_{j=1}^{M_k} \subset \tilde{\mathcal{F}}$ such that

$$\mu\big(A \setminus \bigcup_{j=1}^{M_k} B_k\big) \leq \theta^k \mu(A).$$

Let $k \to \infty$ and set $\mathcal{G} = \{B_j\}_{j=1}^{\infty}$ (Note $\theta < 1$). Then this subcollection of $\mathcal{F}$ satisfies the result. □

Hausdorff Measure and Densities

Definition 26. *Let (X, d) be a metric space, $m \in \mathbb{R}^+$ (m is a non-negative real number), and $\delta > 0$. For $A \subset X$, define*

$$\mathcal{H}^m_\delta(A) = \inf\left\{\sum_i \omega_m \left(\frac{diam\ C_i}{2}\right)^m : A \subset \bigcup_{i=1}^{\infty} C_i \text{ and } diam\ C_i \leq \delta\right\}.$$

*Then the **m-dimensional Hausdorff measure** is given by*

$$\mathcal{H}^m(A) = \lim_{\delta \to 0} \mathcal{H}^m_\delta(A).$$

Remark 27.
- *If $m \in \mathbb{N}$, then $\omega_m = L^m(B_1(0))$ where $L^m(B_1(0))$ is Lebesgue measure of the unit ball in $\mathbb{R}^m$.*
- *The Hausdorff measure of a set always exists (it may be $+\infty$) because $\mathcal{H}^m_\delta(A)$ is a decreasing function of δ.*
- *The Hausdorff measure is a regular Borel measure.*

Definition 28. *Let μ be a Borel regular measure in (X, d). Then for any $A \subset X$, define*

(1) ***the upper density**, $\theta^{*,n}(\mu, A, x)$, to be*

$$\theta^{*,n}(\mu, A, x) = \limsup_{r \to 0} \frac{\mu(A \cap B(x, r))}{\omega_n r^n}.$$

Note that this could be $+\infty$.

(2) ***the lower density**, $\theta^n_*(\mu, A, x)$, to be*

$$\theta^n_*(\mu, A, x) = \liminf_{r \to 0} \frac{\mu(A \cap B(x, r))}{\omega_n r^n}$$

where $B(x, r)$ is the closed ball.

*If $\theta^{*,n}(\mu, A, x) = \theta^n_*(\mu, A, x)$, then define the **density**, $\theta^n(\mu, A, a)$, to be $\theta^n(\mu, A, x) = \theta^n_*(\mu, A, x)$.*

Remark 29. *If $A = X$, then we drop the set in the notation (i.e. $\theta^{*,n}(\mu, x)$ and $\theta^n_*(\mu, x)$).*

Some questions:

(1) What do the upper and lower density tell you about the measure μ.

Example (The Four Corner Cantor Set): Let $\mathcal{C}$ denote the four corner Cantor set. Set $\mu = \mathcal{H}^1 \llcorner \mathcal{C}$. Then one can show that for μ a.e. $x \in \mathcal{C}$,

$$0 < \underbrace{\theta^1_*(\mu \llcorner \mathcal{C}, x) < \theta^{*,1}(\mu \llcorner \mathcal{C}, x)}_{\text{never the same}} < \infty.$$

However, there exists a $C_0 > 1$ such that

$$\frac{1}{C_0} \leq \frac{\mu(B(x,r) \cap \mathcal{C})}{2r} \leq C_0.$$

Theorem 30. *(Marstrand) If $\theta^s(\mu \llcorner A, x)$ exists, positive, and finite for $\mu(A) > 0$, then $s \in \mathbb{N}$.*

Remark 31.
- *There is only 1 interesting n.*
- *Each $\theta^{n,*}, \theta^n_*$ are measurable functions.*

Theorem 32. *Let μ be a Borel regular on (X, d) and $t \geq 0$.*

(1) *If $A_1 \subseteq A_2 \subseteq X$ and $\theta^{*,n}(\mu, A_2, x) \geq t$ for all $x \in A_1$, then $t\mathcal{H}^n(A_1) \leq \mu(A_2)$.*
(2) *If $A \subset X$ and $\theta^{*,n}(\mu, A, x) \leq t$ for all $x \in A$, then $\mu(A) \leq 2^n t\mathcal{H}^n(A)$.*

Important Remarks:

- We are not assuming that A, A_1, or A_2 are measurable.
- If $A_1 = A_2 = A$, then from (1) if $\theta^{*,n}(\mu, A, x) \geq t$ for all $x \in A$, then $t\mathcal{H}^n(A) \leq \mu(A)$.
- All differentiation theorems require a covering lemma (including this one).
- All balls, $B(x, r)$, are closed unless otherwise indicated.

Proof. (Proof of Theorem 32, Part (1))
We can assume that $\mu(A_2) < \infty$ and $t > 0$ for otherwise the result is trivial. We can also assume the strict inequality $\theta^{*,n}(\mu, A_2, x) > t$ since to obtain the conclusion of (1) for t equal to a given $t_1 > 0$ it suffices to prove it for each $t < t_1$. By the definition of lim sup, we have that

$$\inf_{\delta > 0} \sup_{r < \delta} \frac{\mu(A_2 \cap B(x,r))}{\omega_n r^n} > t.$$

For every $\delta_0 > 0$ and $x \in \mathbb{R}^n$, there exists $r_k = r_{k,x} < \delta_0$ such that $\mu(A_2 \cap B(x, r_k)) > t\omega_n r_k^n$. Fix $\delta > 0$. Consider the set

$$\mathcal{F}_\delta = \left\{ B(x,r) \,:\, x \in A_1, r < \frac{\delta}{2}, \text{ and } \mu(A_2 \cap B(x,r)) > t\omega_n r^n \right\}.$$

Using the definition of a lim sup, it follows that $\mathcal{F}_\delta$ is a fine cover of A_1 (i.e. the diameters of the balls are bounded by $\frac{\delta}{2}$ and for every $x \in A_1$ and $\epsilon > 0$ there exists a $B \in \mathcal{F}_\delta$ such that diam $B < \epsilon$ and $x \in B$). Applying the corollary to the Vitali covering lemma, there exists a disjoint subcollection $\mathcal{G} \subset \mathcal{F}_\delta$ (where $\{B_i\} = \mathcal{G}$) such that for all $M \in \mathbb{N}$,

$$A_1 \setminus \bigcup_{i=1}^{M} B_i \subset \bigcup_{B \in \mathcal{G} \setminus \{B_i\}_{i=1}^M} \widehat{B}, \tag{33}$$

where $\widehat{B} = B(x, 5r)$. In fact, this disjoint subcollection $\mathcal{G}$ can be assumed to be countable. In fact since μ is Radon and $\mu(A_2) < \infty$, there exists U open such that $A_2 \subset U$ and $\mu(U) < \infty$. Choose balls $B \in \mathcal{G}$ such that $B \subset U$. Using the fact that $\mathcal{G}$ is a disjoint subcollection and

$B \subset U$ where $\mu(U) < \infty$, it follows that $\mathcal{G}$ is actually countable.) Hence, denote $\mathcal{G} = \{B_i\}_{i=1}^{\infty}$. Fix a $M \in \mathbb{N}$. Using the countability and (33),

$$A_1 \subset \left(\bigcup_{i=1}^{M} B_i\right) \cup \left(\bigcup_{i=M+1}^{\infty} \widehat{B}_i\right),$$

so A_1 is covered by these two sets. Moreover since $\mathcal{G} \subset \mathcal{F}_\delta$, $\mu(A_2 \cap B_i) > t\omega_n(\text{radius } B_i)$ and radius $B_i < \frac{\delta}{2}$. Denote $r_i = \text{radius } B_i$. Using the definition of $\mathcal{H}^n_\delta$,

$$\mathcal{H}^n_{5\delta}(A_1) \leq \sum_{i=1}^{M} \omega_n r_i^n + \sum_{i=M+1}^{\infty} \omega_n 5^n r_i^n. \tag{34}$$

Since $B_i \in \mathcal{F}_\delta$, then

$$\sum_{i=1}^{M} \omega_n r_i^n < \frac{1}{t}\sum_{i=1}^{\infty} \mu(A_2 \cap B_i) = \frac{1}{t}\sum_{i=1}^{\infty} (\mu \llcorner A_2)(B_i) = \frac{1}{t}(\mu \llcorner A_2)\left(\bigcup_{i=1}^{\infty} B_j\right) \leq \frac{1}{t}\mu(A_2) < \infty.$$

Let $M \to \infty$ in (34). Then we have

$$\mathcal{H}^n_{5\delta}(A_1) \leq \sum_{i=1}^{\infty} \omega_n r_i^n < \frac{1}{t}\mu(A_2).$$

Next, let $\delta \to 0$. Then $\mathcal{H}^n(A_1) \leq \frac{1}{t}\mu(A_2)$. □

Proof. (Proof of Theorem 32, Part (2))
Like in the proof of Theorem 32 Part (1), assume $\theta^{*,n}(\mu, A, x) < t$ for all $x \in A$ (i.e. otherwise pass to the limit). By the definition of lim sup, we have that

$$\inf_{\delta>0} \sup_{r<\delta} \frac{\mu(A_2 \cap B(x,r))}{\omega_n r^n} < t.$$

Hence, there exists a δ_x such that for all $r < \delta_x$, $\mu(A \cap B(x,r)) < t\omega_n r^n$. Define

$$A_k = \{x \in A : \mu(A \cap B(x,r)) < t\omega_n r^n \text{ for all } 0 < r < \tfrac{1}{k}\}.$$

Note that $A_k \subset A_{k+1}$ and $A = \bigcup_{k=1}^{\infty} A_k$. Using that μ is a Borel regular measure, and Theorem 8 we have that

$$\mu(A) = \lim_{k\to\infty} \mu(A_k) \qquad \text{(remember } A \text{ may not be measurable).}$$

Let $\delta \in (0, \frac{1}{4k})$ and let $\{C_i\}_{i=1}^{\infty}$ be a countable cover for A_k (i.e. $A_k \subset \bigcup C_i$) such that diam $C_i \leq \delta$. For $x_i \in A_k \cap C_i$, $C_i \subset B(x_i, \text{diam } C_i)$ where diam $C_i < \frac{1}{4k}$. By the definition of A_k,

$$\mu(C_i \cap A_k) \leq \mu(B(x_i, \text{diam } C_i) \cap A_k) < t\omega_n(\text{diam } C_i)^n \leq t\omega_n 2^n \left(\frac{\text{diam } C_i}{2}\right)^n.$$

Hence,

$$\mu(A_k) \leq \sum_{i=1}^{\infty} \mu(C_i \cap A_k) \leq \sum_{i=1}^{\infty} t\omega_n 2^n \left(\frac{\text{diam } C_i}{2}\right)^n.$$

Taking the inf over all such covers $\{C_i\}$, we have by definition of $\mathcal{H}^n_\delta$ that $\mu(A_k) \leq t2^n\mathcal{H}^n_\delta(A_k)$. Let $\delta \to 0$ and conclude that $\mu(A_k) \leq t2^n\mathcal{H}^n(A_k)$. Since $\mathcal{H}^n$ is a regular Borel measure, $\lim_{k\to\infty} \mathcal{H}^n(A_k) = \mathcal{H}^n(A)$. By letting $k \to \infty$, $\mu(A) \leq t2^n\mathcal{H}^n(A)$. □

Applications of the Theorem:

Theorem 35. *Suppose μ is Borel regular measure on (X, d). Let $A \subset X$ such that $\mu(A) < \infty$, A is μ-measurable, and*

$$\theta^{*,n}(\mu, A, x) = 0 \quad \mu \text{ a.e. } x \in A.$$

If $\mathcal{H}^n(A) < \infty$, then

$$\theta^{*,n}(\mathcal{H}^n \llcorner A, x) = 0 \quad \text{for } \mathcal{H}^n \text{ a.e. } x \in X \setminus A.$$

Theorem 36. *Suppose A is a subset of X.*

(1) *If $\mathcal{H}^n(A) < \infty$, then $\theta^{*,n}(\mathcal{H}^n \llcorner A, x) \leq 1$ for $\mathcal{H}^n$ a.e. $x \in A$.*
(2) *If $\mathcal{H}^n_\delta(A) < \infty$ for each $\delta > 0$, then $\theta^{*,n}(\mathcal{H}^n_\infty \llcorner A, x) \geq \frac{1}{2^n}$ for $\mathcal{H}^n$ a.e. $x \in A$.*

Complete proofs can be found in *Lectures on Geometric Measure Theory* by Leon Simon; theorems 3.5 and 3.6 (pg. 14-17).

Corollary 37. *If $0 < \mathcal{H}^n(A) < \infty$ and A a $\mathcal{H}^n$-measurable subset of X, then*

$$\frac{1}{2^n} \leq \theta^{*,n}(\mathcal{H}^n \llcorner A, x) \leq 1.$$

Proof. This follows directly from Theorem 36 since $\mathcal{H}^n \geq \mathcal{H}^n_\delta \geq \mathcal{H}^n_\infty$. □

4. Lecture 4: April 9, 2014

Definition 38. *Let $x \in \mathbb{R}^n$ and μ and ν Radon measures in $\mathbb{R}^n$. Denote*

$$spt\ \mu = \{x : \mu(B(x,r)) > 0 \text{ for all } r > 0\}.$$

(1) *The **upper density** is*

$$\overline{D}_\mu \nu(x) = \begin{cases} \limsup_{r\to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } x \in spt\ \mu \\ +\infty & \text{otherwise} \end{cases}$$

(2) *The **lower density** is*

$$\underline{D}_\mu(\nu(x)) = \begin{cases} \liminf_{r\to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } x \in spt\ \mu \\ +\infty & \text{otherwise} \end{cases}$$

If $\overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x)$, then define $D_\mu \nu(x) := \overline{D}_\mu \nu(x)$ to be the density. Notice that $\overline{D}_\mu \nu$ and $\underline{D}_\mu \nu$ are Borel measurable.

Lemma 39. *For $0 < \alpha < \infty$,*

(1) *If $A \subset \{x : \underline{D}_\mu \nu(x) \le \alpha\}$, then $\nu(A) \le \alpha\mu(A)$*
(2) *If $A \subseteq \{x : \overline{D}_\mu \nu(x) \ge \alpha\}$, then $\nu(A) \ge \alpha\mu(A)$.*

Proof. For a complete proof, see *Lectures on Geometric Measure Theory* by Leon Simon page 26. The proof is similar to the proof of Theorem 32 but use Besicovitch's covering lemma instead of Vitali's covering lemma. □

Theorem 40. *Let μ and ν be Radon measures in $\mathbb{R}^n$. Then $D_\mu \nu$ exists and is finite μ-a.e and is μ-measurable.*

Proof. We can assume without loss of generality that $\mu(\mathbb{R}^n) < \infty$ and $\nu(\mathbb{R}^n) < \infty$ since μ and ν are Radon measures and $\mathbb{R}^n$ is locally compact and separable.

Define $I = \{x : \overline{D}_\mu \nu(x) = +\infty\} \subset \{x : \overline{D}_\mu \nu(x) \ge \alpha\}$. By Lemma (39) for all $\alpha > 0$, $\alpha\mu(I) \le \nu(I)$. Hence, $\mu(I) \le \frac{1}{\alpha}\nu(\mathbb{R}^n) \to 0$ as $\alpha \to \infty$. As a result, $\mu(I) = 0$.

Let $0 < a < b < \infty$ and $R(a,b) = \{x : \underline{D}_\mu \nu(x) < a < b < \overline{D}_\mu \nu(x)\}$. Note this is a similar idea as proving the Lebesgue differentiation theorem. Next, note that

$$\{\underline{D}_\mu \nu(x) < \overline{D}_\mu \nu(x)\} = \bigcup_{\substack{0<a,b\in\mathbb{Q} \\ a<b}} R(a,b).$$

By Lemma (39), $b\mu(R(a,b)) \le \nu(R(a,b))$ and $\nu(R(a,b)) \le a\mu(R(a,b))$. Since $a < b$, this implies that $\mu(R(a,b)) = 0$. Consequently, we showed that for μ a.e. x, $\underline{D}_\mu \nu(x) = \overline{D}_\mu \nu(x)$ so $D_\mu \nu(x)$ exists μ-a.e. and since $\overline{D}_\mu \nu(x) < \infty$ μ-a.e. then $D_\mu \nu(x)$ is finite μ-a.e.

It remains to show that $D_\mu \nu(x)$ is measurable. To do this, we use the following claim:

Claim 41. *For any $r > 0$ and $x_i \to x$, the $\limsup_{i\to\infty} \mu(B(x_i, r)) \le \mu(B(x,r))$.*

Proof. (Proof of Claim) Let $f_i = \chi_{B(x_i,r)}$ and $f = \chi_{B(x,r)}$. We claim that $\limsup_{i\to\infty} f_i(y) \leq f(y)$. First, lim sup can only take the values 0 or 1. Next, there is only one case to consider for if $\limsup_{i\to\infty} f_i(y) = 0$, then the result holds trivially. Suppose $\limsup_{i\to\infty} f_i(x) = 1$. Then for all k there exists a j_k such that $f_{j_k}(y) = 1$ and $|x - x_{j_k}| < \epsilon$. In particular, $y \in B(x_{j_k}, r)$. Suppose on way to a contradiction, $f(y) = 0$. Hence, $y \notin B(x, r)$. Since $B(x, r)$ is closed, there exists an $s > 0$, such that $B(y, s) \subset B(x, r)^c$. Thus, $|x - y| \geq r + s$, but

$$|x - y| \leq |y - x_{j_k}| + |x_{j_k} - x| < r + \epsilon.$$

By choosing ϵ small, we get a contradiction. Therefore, $\limsup_{i\to\infty} f_i(y) \leq f(y)$. This implies that $\liminf_{i\to\infty}(1 - f_i(y)) \geq 1 - f \geq 0$. By Fatou's Lemma,

$$\int_{B(x,2r)} (1 - f)\, d\mu \leq \liminf_{i\to\infty} \int_{B(x,2r)} (1 - f_j)\, d\mu.$$

As a result,

$$\mu(B(x, 2r)) - \mu(B(x, r)) \leq \mu(B(x, 2r)) - \limsup_{i\to\infty} \mu(B(x_i, r)).$$

Because $\mu(B(x, 2r)) < \infty$ for we assumed the measure of the whole space was finite, $\limsup_{i\to\infty} \mu(B(x_i, r)) \leq \mu(B(x, r))$. □

A function f, defined almost everywhere on $\mathbb{R}^n$ is μ-measurable if and only if there exists a sequence of simple $\mathcal{A}$-functions which converge to f almost everywhere where $\mathcal{A}$ is the σ-algebra for μ. Thus, we allow f to be undefined on a null set of μ. By the definition of upper semi-continuous and Claim 41, $x \mapsto \mu(B(x, r))$ and $x \mapsto \nu(B(x, r))$ are an upper semi-continuous functions. Recall that upper (lower) semi-continuous functions are Borel measurable functions. Since μ is a Radon measure, it contains the Borel sets and so these functions are μ-measurable. Hence, $x \mapsto \frac{\nu(B(x,r))}{\mu(B(x,r))}$ is a μ-measurable function for each fixed r. Note that we assume the balls are non-degenerate and have finite measure. Since $D_\mu\nu(x) = \lim_{r\to 0} \frac{\nu(B(x,r))}{\mu(B(x,r)}$ exists for μ-a.e. $x \in \mathbb{R}^n$, the definition of μ-measurable gives that $D_\mu\nu(x)$ is μ-measurable. □

Differentiation Theorem for Radon Measures

Theorem 42. *Let μ and ν Radon measures with $\nu \ll \mu$. Then*

$$\nu(A) = \int_A D_\mu\nu\, d\mu \quad \textit{for all } A\ \mu-\textit{measurable.} \tag{43}$$

***Radon-Nikodym**: For any μ and ν Radon, there exists ν_{ac} and ν_s such that $\nu = \nu_{ac} + \nu_s$, $\nu_{ac} \ll \mu$, and $\nu_s \perp \mu$. In addition, $D_\mu\nu_{ac} = D_\mu\nu$ and $D_\mu\nu_s = 0$ for μ-a.e. x and*

$$\nu(A) = \int_A D_\mu\nu\, d\mu + \nu_S(A).$$

Proof. (Proof of (43)) Let $Z = \{x : D_\mu\nu(x) = 0\}$ and $I = \{x : D_\mu\nu(x) = +\infty\}$. In the previous proof, we showed that $\mu(I) = 0$. Since $\nu \ll \mu$, $\nu(I) = 0$.

Note that $Z \subset \{x : D_\mu \nu(x) < \alpha\}$ for all $\alpha > 0$. By Lemma (39), $\nu(Z) \leq \alpha\mu(Z)$ for all $\alpha > 0$. Letting $\alpha \to 0$ (and assuming $\mu(Z) < \infty$), then $\nu(Z) \to 0$.

Let A be μ-measurable and $1 < t < \infty$. The idea is to decompose A into sets where the $D_\mu \nu(x)$ is almost constant. Let $m \in \mathbb{Z}^+ > 0$. Define

$$A_m = A \cap \{x : t^m \leq D_\mu \nu(x) < t^{m+1}\}$$

and so

$$A \setminus \bigcup_{m=-\infty}^{\infty} A_m \subset (Z \cup I \cup \{\overline{D}_\mu \nu \neq \underline{D}_\mu \nu\})$$

From above, we showed that $\mu(Z) = \mu(I) = 0$. Moreover from Theorem (42), we showed in the proof that $\mu(\{\overline{D}_\mu \nu \neq \underline{D}_\mu \nu\}) = 0$. Hence, $\mu\left(A \setminus \bigcup_{m=-\infty}^{\infty} A_m\right) = 0$, which implies that $\nu\left(A \setminus \bigcup_{m=-\infty}^{\infty} A_m\right) = 0$ ($\nu \ll \mu$). Because the A_m's are disjoint we have

$$\begin{aligned}
\nu(A) = \sum_{-\infty}^{\infty} \nu(A_m) \leq \sum_{-\infty}^{\infty} t^{m+1} \mu(A_m) = t \sum_{-\infty}^{\infty} t^m \mu(A_m) \qquad & \text{by Lemma (39)} \\
&= t \sum_{-\infty}^{\infty} \int_{A_m} t^m \, d\mu \\
&= t \sum_{-\infty}^{\infty} \int_{A_m} D_\mu \nu \, d\mu \\
&= t \int_A D_\mu \nu \, d\mu.
\end{aligned}$$

On the other hand, using the definition of A_m

$$\nu(A) \geq \sum_{-\infty}^{\infty} t^m \mu(A_m) = \frac{1}{t} \sum_{-\infty}^{\infty} \int_{A_m} t^{m+1} \, d\mu \geq \frac{1}{t} \int_A D_\mu \nu \, d\mu.$$

Hence,

$$\frac{1}{t} \int_A D_\mu \nu \, d\mu \leq \nu(A) \leq t \int_A D_\mu \nu \, d\mu.$$

As this is true for all $1 < t < \infty$, letting $t \to 1$, gives the desired equation. □

Proof. (Proof of Radon-Nikodym) Assume $\mu(\mathbb{R}^n) < \infty$ and $\nu(\mathbb{R}^n) < \infty$. Define

$$\mathcal{E} = \{A \in \mathbb{R}^n : A \text{ Borel}, \mu(\mathbb{R}^n \setminus A) = 0\}.$$

Choose B_k Borel such that $B_k \in \mathcal{E}$ and

$$\nu(B_k) \leq \inf_{A \in \mathcal{E}} \nu(A) + \tfrac{1}{k}.$$

Set $B = \bigcap_k B_k$. We want to show that B is in $\mathcal{E}$. By countable subadditivity, $\mu(\mathbb{R}^n \setminus B) \leq \sum_k \mu(\mathbb{R}^n \setminus B_k) = 0$ since $B_k \in \mathcal{E}$. Hence, $B \in \mathcal{E}$.

By monotonicity, $\nu(B) \leq \nu(B_k) \leq \inf_{A \in \mathcal{E}} \nu(A) + \frac{1}{k}$. As this is true for all k, let $k \to \infty$ to conclude that $\nu(B) = \inf_{A \in \mathcal{E}} \nu(A)$.

Define

$$\nu_{ac} = \nu \llcorner B \quad \text{and} \quad \nu_s = \nu \llcorner (\mathbb{R}^n \setminus B).$$

First, it is clear that $\nu = \nu_{ac} + \nu_s$ by definition. To see that $\nu_s \perp \mu$. Let B and $\mathbb{R}^n \setminus B$ be the disjoint subsets. By definition of ν_s, it is clear that $\nu_s(A) = \nu_s(A \cap (\mathbb{R}^n \setminus B))$ for all $A \in \mathbb{R}^n$. On the other hand, since $B \in \mathcal{E}$, $\mu(\mathbb{R}^n \setminus B) = 0$. Hence,

$$\mu(A) = \mu(A \cap B) + \underbrace{\mu(A \cap (\mathbb{R}^n \setminus B))}_{\text{set of measure 0}} = \mu(A \cap B).$$

It remains to show that $\nu_{ac} \ll \mu$. Suppose on way to a contradiction that $\mu(A) = 0$ and $\nu_{ac}(A) > 0$. One sees that $\mu(\mathbb{R}^n \setminus (B \setminus A)) \leq \mu(\mathbb{R}^n \setminus B) + \mu(A) = 0$, so $B \setminus A \in \mathcal{E}$. However, $\nu_{ac}(B \setminus A) < \nu_{ac}(B) = \nu(B)$. This is a contradiction for $\nu(B)$ was the infimum. Therefore, $\nu_{ac} \ll \mu$.

From the previous proof and $\nu_{ac} \ll \mu$,

$$\nu_{ac}(A) = \int_A D_\mu \nu_{ac}\, d\mu \quad \Rightarrow \quad \nu(A) = \int_A D_\mu \nu_{ac}\, d\mu + \nu_s(A).$$

We want $D_\mu \nu_{ac} = D_\mu \nu$. Since the supports of ν_{ac} and ν_s are different, $D_\mu \nu = D_\mu(\nu_{ac} + \nu_s) = D_\mu \nu_{ac} + D_\mu \nu_s$. Hence, it suffices to show that $D_\mu \nu_s$ is 0 μ a.e. Set

$$C_\alpha = \{x \in B \,:\, D_\mu \nu_s > \alpha\},$$

which implies by Lemma (39), $\nu_s(C_\alpha) \geq \alpha \mu(C_\alpha)$. However since ν_s is supported on $\mathbb{R}^n \setminus B$ and C_α is a subset of B, $\nu_s(C_\alpha) = 0$ and so $\mu(C_\alpha) = 0$. This is true for every $\alpha > 0$ and so it follows that $D_\mu \nu_s$ is 0 for μ a.e. □

In this section, X is always assumed to be locally compact and separable metric space.

Background: From real analysis, the *Riesz Representation Theorem* says:
Let (X, d) be a locally compact and separable metric space.

(1) Let $L : C_c^+(X, \mathbb{R}) \to \mathbb{R}$ be a bounded positive linear functional (i.e. if $f \in C_c(X, \mathbb{R})$ and $f \geq 0$, then $L(f) \geq 0$). Then there exists a unique (finite) Radon measure μ such that $Lf = \int f\, d\mu$.
(2) If $T : C_c(X, \mathbb{R}) \to \mathbb{R}$ is a bounded linear functional (i.e. $|Tf| \leq C||f||_\infty$), then there exists a signed Radon measure μ such that $Tf = \int f\, d\mu$.

From Lecture 1, recall that we proved the following theorem about Borel regular measures: Let μ be Borel regular on (X, d). Suppose $X = \bigcup_{j=1}^\infty V_j$, V_j open, and $\mu(V_j) < \infty$. Then

(1) $\mu(A) = \inf_{\substack{A \subset U \\ U \text{ open}}} \{\mu(U)\}$ for all $A \subset A$.
(2) $\mu(A) = \sup_{\substack{C \subset A \\ C \text{ closed}}} \{\mu(C)\}$ for all A μ-measurable.

Important Remark: The finiteness of Radon measures μ on compact sets allows us to integrate functions which are compactly supported. In fact, let $C_c(X, \mathbb{R}^m)$ be the set of compactly supported continuous functions from $X \to \mathbb{R}^m$, μ a Radon measure, and a μ-measurable function $v : X \to \mathbb{R}^m$ such that $|v| = 1$ for μ a.e. x. Then we can associate with

each v and μ, a linear functional $L : C_c(X, \mathbb{R}^m) \to \mathbb{R}$ defined by

$$Lf = \int \langle f, v \rangle \, d\mu,$$

where $\langle \cdot, \cdot \rangle$ is the inner product. If supp $f \subset K$ where K is compact, then

$$|Lf| \leq ||f||_\infty \mu(K).$$

Hence, if you fix a compact set, this linear functional is bounded.

Theorem 44. *(Riesz Representation Theorem)*
Suppose X is locally compact and separable metric space. Let $L : C_c(X, \mathbb{R}^m) \to \mathbb{R}$ be a linear functional such that for all $K \subset X$, K compact, and

$$\sup\{Lf : f \in C_c(X, \mathbb{R}^m), ||f|| \leq 1, \text{ and spt } f \subset K\} < \infty.$$

Then there exists a unique Radon measure μ and a μ-measurable function $v : X \to \mathbb{R}^m$ such that $|v| = 1$ for μ-a.e. x and

$$Lf = \int \langle f, v \rangle \, d\mu \qquad \text{for all } f \in C_c(X, \mathbb{R}^m).$$

5. Lecture 5: April 14, 2014

In this section, we give a proof of the Riesz Representation Theorem, the main strategy for which is as follows: define a set function μ on X on open subsets $V \subset X$ by writing

$$\mu(V) := \sup\{Lf : f \in C_c(V, \mathbb{R}^m) \text{ with } |f| \leq 1\}$$

extended to arbitrary $A \subset X$ by

$$\mu(A) := \inf\{\mu(U) : A \subset U \text{ and } U \text{ is open } \}$$

Additionally, let $C_c^+(X)$ be the nonnegative functions of $C_c(X, \mathbb{R})$ and write for $f \in C_c^+(X)$

$$\lambda(f) := \sup\{|Lg| : g \in C_c(X, \mathbb{R}^m) \text{ and } |g| \leq f\}$$

We will prove the theorem in steps by showing:

(1) μ is a Radon measure,
(2) if $f_1, f_2 \in C_c^+(X)$ then $\lambda(f_1 + f_2) = \lambda(f_1) + \lambda(f_2)$,
(3) if $f \in C_c^+(X)$ then $\lambda(f) = \int_X f d\mu$,
(4) there exists $\nu : X \to \mathbb{R}^m$ μ-measurable such that for $g \in C_c(X, \mathbb{R}^m)$, we have

$$Lg = \int_X \langle g, \nu \rangle d\mu$$

(5) $|\nu(x)| = 1$ for μ-a.e. $x \in X$.

Proof. (Proof of Riesz Representation Theorem) Noting that $\mu(\emptyset) = 0$ follows from the definition ($f \equiv 0$ provides the function). We prove that μ is a measure using the definition. We first show countable subadditivity relative to open sets and then we extend it to arbitrary sets.

Let $V = \bigcup_{i \geq 1} V_i$ with each V_i open, and let $g \in C_c(X, \mathbb{R}^m)$ be supported in V with $|g| \leq 1$. As the V_i cover V, they cover the support of g, and thus for some k sufficiently large, we have

$$\text{spt}(g) \subset \bigcup_{i=1}^k V_i$$

Let $\{\xi_i\}_{i=1}^k$ be a partition of unity subordinate to $V_1, \ldots, V_k$, so that

$$\xi_i \in C_c(V_i, \mathbb{R}^m) \quad \text{with} \quad 0 \leq \xi_i \leq 1 \quad \text{and} \quad \sum_{i=1}^k \xi_i(x) = 1 \text{ for } x \in \text{spt}(g)$$

noting that this gives $g \equiv 0$ on $(\cup_{i=1}^k V_i)^c$ and $g = \sum \xi_i g$. Using this and the definition of μ to estimate Lg, we obtain

$$Lg = \sum_{i=1}^k L(\xi_i, g) \leq \sum_{i=1}^k \mu(V_i) \leq \sum_{i=1}^\infty \mu(V_i)$$

so that taking the supremum over $g \in C_c(V, \mathbb{R}^m)$ with $|g| \leq 1$ gives

$$\mu(V) \leq \sum_{i=1}^\infty \mu(V_i)$$

Now, for an arbitrary $A \subset X$, let $A \subset \bigcup_{i=1}^\infty A_i$. For each $\epsilon > 0$, we can choose a sequence of open sets U_i with $\mu(U_i) \leq \mu(A_i) + 2^{-i}\epsilon$, so that as μ is monotonic (by definition)

$$\mu(A) \leq \mu(\bigcup U_i) \leq \sum_{i=1}^\infty \mu(U_i) \leq \epsilon + \sum_{i=1}^\infty \mu(A_i),$$

using subadditivity relative to open sets. As ϵ is arbitrary, we let it go to 0 and obtain the desired inequality, showing that μ is sub additive and thus a measure.

Now we show it is Borel regular, using Caratheodory's criterion. Let V_1, V_2 be open sets with $d(V_1, V_2) > 0$, and take $g_i \in C_c(V_i, \mathbb{R}^m)$ with $|g_i| \leq 1$. As $|g_1 + g_2| \leq 1$ by the triangle inequality, we have

$$L(g_1) + L(g_2) = L(g_1 + g_2) \leq \mu(V_1 \cup V_2)$$

by the definition of μ, and so taking the supremum over g_1 and g_2 gives

$$\mu(V_1) + \mu(V_2) \leq \mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2)$$

by subadditivity, and it follows that μ is additive on metrically separated open sets. If $d(A, B) > 0$ there exist V_A and V_B open sets such that $A \subset V_A$, $B \subset V_B$ and $d(V_A, V_B) > 0$. Using the corresponding result for metrically separated open sets we prove that μ is additive on metrically separated Borel sets thus μ is Borel regular. Finally, we show that μ is finite on compact sets, and therefore Radon: as X is locally compact, for every $x \in X$, there is an $r_x > 0$ such that $B(x, r_x)$ has compact closure. Fixing $K \subset X$ compact, we have

$$K \subset \bigcup_{x \in K} B(x, r_x) \quad \text{ and so } \quad K \subset \bigcup_{i=1}^{n} B(x_i, r_{x_i})$$

for some finite collection $x_1, \ldots, x_n \in K$ by compactness. Using subadditivity and the definition of μ on open sets, we obtain

$$\mu(K) \leq \sum_{i=1}^{n} \mu(\cup B(x_i, r_{x_i})) \leq \sum_{i=1}^{n} \sup\{Lg : |g| \leq 1 \text{ and } \operatorname{spt}(g) \subset B(x_i, r_{x_i})\} < \infty$$

by assumption on L. Thus, μ defines a Radon measure on X, proving claim (1)

Now, letting λ be as above, so that for $f \in C_c^+(X)$

$$\lambda(f) = \sup\{|Lg| : g \in C_c(X, \mathbb{R}^m) \text{ and } |g| \leq f\}$$

If $f_1 \leq f_2$ with $f_i \in C_c^+(X)$ then we have $|g| \leq f_1 \implies |g| \leq f_2$ and so for every $c \geq 0$

$$\lambda(f_1) \leq \lambda(f_2) \qquad \text{and} \qquad \lambda(cf_1) = c\lambda(f_2)$$

Choose $g_1, g_2 \in C_c(X, \mathbb{R}^m)$ such that $|g_i| \leq f_i$ and this gives

$$|g_1 + g_2| \leq f_1 + f_2$$

and without loss of generality we can assume that Lg_1 and Lg_2 are nonnegative (by replacing them by their negative) and so

$$Lg_1 + Lg_2 = L(g_1 + g_2) = |L(g_1 + g_2)| \leq \lambda(f_1 + f_2) \quad \text{ so that } \quad \lambda(f_1) + \lambda(f_2) \leq \lambda(f_1 + f_2)$$

by taking the supremum over g_1, g_2. Conversely, if $g \in C_c(X, \mathbb{R}^m)$ satisfies $|g| \leq f_1 + f_2$ then for $i = 1, 2$ let

$$g_i = \begin{cases} \frac{f_i}{f_1 + f_2} g & \text{if } f_1 + f_2 > 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{then} \quad |g_i| \leq f_i \text{ and } g = g_1 + g_2$$

so that by definition of λ, we have

$$|Lg| \leq |Lg_1 + Lg_2| \leq |Lg_1| + |Lg_2| \leq \lambda(f_1) + \lambda(f_2)$$

so that taking the supremum over $|g| \leq f_1 + f_2$ we obtain

$$\lambda(f_1 + f_2) \leq \lambda(f_1) + \lambda(f_2)$$

proving claim (2)

Now, fix $f \in C_c^+(X)$ and let $\epsilon > 0$. Let $0 = t_0 < t_1 < \ldots < t_N$ with

$$t_i - t_{i-1} < \epsilon \qquad \frac{3}{2}\|f\|_\infty < t_N < 2\|f\|_\infty \qquad \mu(f^{-1}\{t_i\}) = 0$$

which is possible as f is continuous with compact support and $\mu \circ f^{-1}$ is a finite measure and thus has at most countably many atoms. Let $U_i = f^{-1}\big((t_{i-1}, t_i)\big)$, so that the U_i are open and disjoint sets of finite μ measure, and

$$\mu\big(\mathrm{spt}(f) \setminus \cup_{i=1}^N U_i\big) = 0$$

Now, as μ is inner regular, for all $U \subset X$ open we can write

$$\mu(U) = \sup\{\lambda(h) : h \in C_c^+(U) \text{ and } h \leq 1\}$$

using the definition of μ and moreover, for each $\epsilon > 0$, we can find a compact set $K \subset U$ with $\mu(U \setminus K) < \epsilon$. Combining, we have for $\epsilon > 0$ and each $1 \leq i \leq N$, a function $h_i \in C_c^+(U_i)$ with

$$\mu\big(U_i \setminus h_i^{-1}\{1\}\big) < \frac{\epsilon}{N} \qquad \text{and} \qquad \mu(U_i) \leq \lambda(h_i) + \frac{\epsilon}{N}$$

From the definition of λ,

$$\lambda(h_i) \leq \sup\{|Lg| : g \in C_c(U_i, \mathbb{R}^m) \text{ and } |g| \leq 1\} = \mu(U_i)$$

and so we obtain

$$\lambda(h_i) \leq \mu(U_i) \leq \lambda(h_i) + \frac{\epsilon}{N}$$

Letting $A \subset X$ be the open set

$$A = \{x : f(x)\left(1 - \sum_{i=1}^N h_i(x)\right) > 0\} \qquad \Longrightarrow \qquad \mu(A) \leq \mu(\cup_1^N U_i \setminus h^{-1}\{1\}) \leq N\frac{\epsilon}{N} = \epsilon$$

then from the definition,

$$\lambda(f - f\sum h_i) \leq \sup\{|Lg| : |g| \leq \|f\|_\infty \chi_A\} \leq \|f\|_\infty \mu(A) \leq \|f\|_\infty \epsilon$$

Using the second claim proved above, we can write

$$\lambda(f) = \lambda(f - f\sum_{i=1}^N h_i) + \lambda(f\sum_{i=1}^N h_i) \leq \|f\|_\infty \epsilon + \sum_{i=1}^N \lambda(fh_i)$$

As the h_i are supported in $U_i = \{x : t_{i-1} < f(x) < t_i\}$ it follows that

$$\lambda(f) \leq \|f\|_\infty \epsilon + \sum_{i=1}^N t_i \mu(U_i)$$

and as $f - f\sum h_i \geq 0$, we have

$$\sum_{i=1}^N t_{i-1}\mu(U_i) \leq \lambda(f\sum_{i=1}^N h_i) \leq \lambda(f)$$

so that

$$\sum_{i=1}^N t_{i-1}\mu(U_i) \leq \lambda(f) \leq \|f\|_\infty \epsilon + \sum_{i=1}^N t_i \mu(U_i)$$

Now, as

$$\sum_{i=1}^{N} t_{i-1}\mu(U_i) \leq \int f d\mu \leq \sum_{i=1}^{N} t_i \mu(U_i)$$

we have for all $\epsilon > 0$

$$|\lambda(f) - \int f d\mu| \leq 3\epsilon \|f\|_\infty + \epsilon\mu(\mathrm{spt}(f))$$

and letting $\epsilon \to 0$ shows that if $f \in C_c^+(X)$, we have $\lambda(f) = \int f d\mu$, verifying claim (3).

Now, fix $\mathbf{e} \in \mathbb{S}^{n-1}$ and for $f \in C_c(X)$, let $\lambda_{\mathbf{e}}(f) := L(f\mathbf{e})$, so that

$$|\lambda_{\mathbf{e}}(f)| \leq \sup\{|Lg| : g \in C_c(X, \mathbb{R}^m) \text{ and } |g| \leq |f|\} \leq \lambda(|f|) \leq \int |f| d\mu \leq \|f\|_\infty \mu(\mathrm{spt\,f}).$$

Thus $\lambda_{\mathbf{e}}$ is a bounded linear functional on $C_c(U)$ where U open and $\bar{U}$ compact, so that, by Riesz Representation on $C_c(X)$, there is a signed Radon measure φ on X with $\lambda_{\mathbf{e}}(f) = \int f d\varphi$. As $|\varphi| \ll \mu$ the Radon-Nikodym theorem gives us (for each $\mathbf{e} \in \mathbb{S}^{n-1}$) a φ-measurable function $v_{\mathbf{e}} \in L^\infty(X)$ with $\varphi = v_{\mathbf{e}}\mu$. Letting $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be an orthonormal basis of $\mathbb{R}^m$, we define the function $\nu : X \to \mathbb{R}^m$ by $\nu(x) = \sum_1^m v_{\mathbf{e}_j}(x)\mathbf{e}_j$ so that for $g \in C_c(X, \mathbb{R}^m)$ we have

$$L(g) = L(\sum_{j=1}^{m} \langle g, \mathbf{e}_j\rangle \mathbf{e}_j) = \sum_{j=1}^{m} \int \langle g, \mathbf{e}_j\rangle v_{\mathbf{e}_j} d\mu = \int \langle g, \nu\rangle d\mu,$$

giving claim (4). We complete the proof by showing that $|\nu(x)| = 1$ for μ-a.e. $x \in X$.

If U is an open subset of X with $\mu(U) < \infty$, then by definition

$$\mu(U) = \sup\left\{Lg = \int \langle g, \nu\rangle d\mu : g \in C_c(U, \mathbb{R}^m) \text{ and } |g| \leq 1\right\}$$

Choosing a sequence of functions $f_k \in C_c(U, \mathbb{R}^m)$ such that $|f_k| \leq 1$ and $f_k\nu \to |\nu|$ μ-a.e., Lebesgue dominated convergence gives

$$\int_U |\nu| d\mu = \lim_{k\to\infty} \int \langle f_k, \nu\rangle d\mu \leq \mu(U)$$

and moreover, as $|f| \leq 1$ we have $\langle f, \nu\rangle \leq |\nu|$ and as μ is a nonnegative measure, this gives

$$\int \langle f, \nu\rangle d\mu \leq \int_U |\nu| d\mu \quad \Longrightarrow \quad \mu(U) \leq \int_U |\nu| d\mu$$

so that combining the above gives $|\nu| = 1$ μ-a.e., completing the proof. □

6. Lecture 6: April 16, 2014

Differentiation of Lipschitz Functions

Definition 45.
*Let $A \subset \mathbb{R}^n$, and $f : A \to \mathbb{R}^m$. We say f is **Lipschitz** if there is an $L > 0$ so that*

$$|f(x) - f(y)| \leq L|x - y| \quad \text{for } x, y \in A \qquad \text{and} \qquad Lip(f) := \sup_{x \neq y \in A} \frac{|f(x) - f(y)|}{|x - y|}$$

*is the Lipschitz constant of f. We say f is **locally Lipschitz** provided for every compact set $K \subset A$, the function $f|_K$ is Lipschitz; here, the (local) Lipschitz constant may depend on K.*

The following theorem shows that we can assume $A = \mathbb{R}^n$ without any loss of generality.

Theorem 46 (Extension of Lipschitz Functions)**.**
If $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ is Lipschitz, then there exists $\overline{f} : \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz with

$$Lip(\overline{f}) \leq \sqrt{m} Lip(f) \qquad \text{and} \qquad \overline{f}|_A = f$$

Proof. First, assume $f : A \to \mathbb{R}$ ($m = 1$) and write $L := \mathrm{Lip}(f)$. For $x \in \mathbb{R}^n$ define

$$\overline{f}(x) := \inf\{f(a) + L|x - a| : a \in A\}$$

so that $\overline{f}|_A \leq f$ by definition. For every $a \in A$, we have

$$f(b) - f(a) \leq |f(b) - f(a)| \leq L|b - a| \quad \implies \quad f(b) \leq f(a) + L|b - a|$$

so that, taking the infimum over $a \in A$ we obtain $f \leq \overline{f}|_A$, and $\overline{f}$ extends f, as desired. Now,

$$\overline{f}(x) = \inf_{a \in A}\{f(a) + L|x - a|\} \leq \inf_{a \in A}\{f(a) + L|y - a|\} + L|x - y| = \overline{f}(y) + L|x - y|$$

so that $\overline{f}$ is Lipschitz with $\mathrm{Lip}(\overline{f}) = L$.

For $f : A \to \mathbb{R}^m$ Lipschitz, the coordinate functions $f_j := \langle f, \mathbf{e}_j \rangle : A \to \mathbb{R}$ and Lipschitz, and so we apply the above argument to the components f_j to obtain the general result. □

The following theorem allows us to extend f without increasing the Lipschitz constant, and is stated without proof.

Theorem 47 (Kriszbraun)**.**
If $f : A \to \mathbb{R}^m$ is Lipschitz, it has a Lipschitz extension $\overline{f} : \mathbb{R}^n \to \mathbb{R}^m$ with $Lip(\overline{f}) = Lip(f)$.

We now discuss differentiability for Lipschitz functions.

Definition 48. *The function $f : \mathbb{R}^n \to \mathbb{R}^m$ is **differentiable at** x if there exists $L : \mathbb{R}^n \to \mathbb{R}^m$ linear such that*

$$\lim_{y \to x} \frac{|f(y) - f(x) + L(x - y)|}{|x - y|} = 0 \quad \text{or equivalently} \quad f(y) = f(x) + L(y - x) + o(|x - y|)$$

If such an L exists it is unique, and we write $L = Df(x)$ for the derivative of f at x.

Theorem 49 (Rademacher)**.**
If $f : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz, then f is differentiable $\mathcal{H}^n$-ae.

Proof. By Kriszbraun's or the Lipschitz extension theorem, it suffices to assume that f is Lipschitz on $\mathbb{R}^n$, and as a function into $\mathbb{R}^n$ is differentiable if and only if its components are differentiable, we may assume without loss of generality that $m = 1$; let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz with $L = \mathrm{Lip}(f)$.

For each $\nu \in \mathbb{S}^{n-1}$, define

$$D_\nu f(x) := \lim_{t\to 0} \frac{f(x+t\nu) - f(x)}{t}$$

We claim that $D_\nu f(x)$ exists for $\mathcal{H}^n$-ae $x \in \mathbb{R}^n$. Writing

$$\overline{D_\nu} f(x) := \limsup_{t\to 0} \frac{f(x+t\nu) - f(x)}{t} \quad \text{and} \quad \underline{D_\nu} f(x) := \liminf_{t\to 0} \frac{f(x+t\nu) - f(x)}{t}$$

then f locally Lipschitz implies both $|\overline{D_\nu} f|$ and $|\underline{D_\nu} f|$ are bounded by L. Let

$$A_\nu := \{x \in \mathbb{R}^n : D_\nu f(x) \text{ does not exist}\} = \{x \in \mathbb{R}^n : \underline{D_\nu} f(x) < \overline{D_\nu} f(x)\}$$

Now, as the map $t \mapsto f(x+t\nu)$ is Lipschitz from $\mathbb{R} \to \mathbb{R}$, it is of bounded variation and thus is differentiable $\mathcal{H}^n$-ae. Thus for any line $\mathcal{L}$ parallel to ν, we have $\mathcal{H}^1(\mathcal{L} \cap A_\nu) = 0$, so that integrating over the lines parallel to ν, we obtain $\mathcal{H}^n(A_\nu) = 0$, and $D_\nu f(x)$ exists $\mathcal{H}^n$-ae, verifying the claim.

Next, let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be an orthonormal basis of $\mathbb{R}^n$, and write

$$\frac{\partial f}{\partial x_i} := D_{\mathbf{e}_i} f \qquad \text{and} \qquad \mathrm{grad}(f) := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

We claim that for $\nu \in \mathbb{S}^{n-1}$, we have

$$D_\nu f(x) = \langle \mathrm{grad}(f), \nu \rangle$$

Take $\xi \in C_c^\infty(\mathbb{R}^n)$, and observe that by change of variables, one has

$$\int_{\mathbb{R}^n} \frac{f(x+t\nu) - f(x)}{t} \xi(x) dx = -\int_{\mathbb{R}^n} f(x) \frac{\xi(x) - \xi(x-t\nu)}{t} dx$$

As the difference quotients $\frac{1}{t}(f(x+t\nu) - f(x)) \to D_\nu f(x) \leq L < \infty$, we let $t \to 0$ in the above and apply dominated convergence, obtaining

$$\int_{\mathbb{R}^n} D_\nu f(x) \xi(x) dx = -\int_{\mathbb{R}^n} f(x) D_\nu \xi(x) dx = -\int_{\mathbb{R}^n} f(x) \langle \mathrm{grad}(\xi), \nu \rangle dx$$

as ξ is a smooth function. Continuing the above equalities, we obtain

$$\int_{\mathbb{R}^n} D_\nu f(x) \xi(x) dx = -\sum_{i=1}^n \left(\int_{\mathbb{R}^n} f(x) \frac{\partial \xi}{\partial x_i}(x) dx \right) \langle \mathbf{e}_i, \nu \rangle = \sum_{i=1}^n \int_{\mathbb{R}^n} D_{\mathbf{e}_i} f(x) \langle \mathbf{e}_i, \nu \rangle \xi(x) dx$$

Which gives us

$$\int_{\mathbb{R}^n} D_\nu f(x) \xi(x) dx = \int_{\mathbb{R}^n} \langle \mathrm{grad}(f), \nu \rangle \xi(x) dx$$

and so $D_\nu f(x) = \langle \mathrm{grad}(f)(x), \nu \rangle$ for a.e. $x \notin A_\nu$. As the set of x where this holds depends on ν, however, we cannot complete the proof until we show that there is a set of full measure for which the directional derivatives exist and for which the desired identity holds.

As $\mathbb{S}^{n-1}$ is separable, let $\mathcal{D} = \{\nu_k\}$ be a countable dense subset of $\mathbb{S}^{n-1}$, and write

$$A = \bigcap_{k\geq 1} A_k \qquad \text{where} \qquad A_k := \{x \in \mathbb{R}^n : \mathrm{grad}(f)(x) \text{ exists, and } D_{\nu_k} f(x) = \langle \mathrm{grad}(f)(x), \nu_k \rangle\}$$

and note that

$$\mathcal{H}^n(\mathbb{R}^n \setminus A) \leq \sum_{k=1}^{\infty} \mathcal{H}^n(\mathbb{R}^n \setminus A_k) = 0$$

We complete the proof by showing that for all $\nu \in \mathcal{S}^{n-1}$ and $x \in A$, f is differentiable at x and $D_\nu f(x) = \langle \text{grad}(f)(x), \nu \rangle$. For $x \in A$, $t \neq 0 \in \mathbb{R}$ and $\nu \in \mathbb{S}^{n-1}$, write

$$Q(x, \nu, t) := \frac{f(x + t\nu) - f(x)}{t} - \langle \text{grad}(f), \nu \rangle$$

As $\mathcal{D}$ is dense in $\mathbb{S}^{n-1}$, for every $\epsilon > 0$ and $\nu \in \mathbb{S}^{n-1}$ there is a k with

$$|\nu - \nu_k| < \epsilon' := \frac{\epsilon}{2(1 + \sqrt{n})L}$$

As $x \in A$, we have $Q(x, \nu_k, t) \to 0$ as $t \to 0$, so that for each $\nu_k \in \mathcal{D}$ there is a $\delta_k > 0$ with

$$|t| < \delta_k \quad \Longrightarrow \quad |Q(x, \nu_k, t)| < \frac{\epsilon}{2}$$

As $\mathbb{S}^{n-1}$ is compact and $\mathcal{D}$ is dense, there is $N < \infty$ such that $\nu_1, \ldots, \nu_N$ form a ϵ' separated net in $\mathbb{S}^{n-1}$; let $\delta = \min_{1 \leq k \leq N} \delta_k > 0$.

Observe that for every $\nu' \in \mathbb{S}^{n-1}$ we have

$$|Q(x, \nu, t) - Q(x, \nu', t)| \leq (1 + \sqrt{n})L|\nu - \nu'|$$

Thus, if $x \in A$, $\nu \in \mathbb{S}^{n-1}$ and $|t| < \delta$, take $k < N$ with $|\nu_k - \nu| \leq \epsilon'$ and

$$|Q(x, \nu, t)| \leq |Q(x, \nu_k, t)| + |Q(x, \nu_k, t) - Q(x, \nu, t)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and so $Q(x, \nu, t) \to 0$ as $t \to 0$ for all $x \in A$ and $\nu \in \mathbb{S}^{n-1}$, as desired. □

Thus, for $f : \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz, we can write

$$f(x + t\nu) = f(x) + t\langle \text{grad}(f)(x), \nu \rangle + o(|t|)$$

For $y \neq x \in \mathbb{R}^n$, let $\nu = \frac{y-x}{|y-x|}$ and take $t = |y - x|$. Applying the above gives

$$f(y) = f(x) + \langle \text{grad}(f)(x), y - x \rangle + o(|y - x|)$$

and so $Df(x) = \text{grad}(f)(x)$.

Theorem 50. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz. For all $\epsilon > 0$ there exists a $g \in C^1(\mathbb{R}^n)$ such that*

$$\mathcal{H}^n\big(\{x \in \mathbb{R}^n : f(x) \neq g(x) \text{ and } Df(x) \neq Dg(x)\}\big) < \epsilon$$

Proof. Whitney's Extension Theorem □