Math 526

Homework due 06/04/2014

Reading from Folland 2nd edition: Chapter 8: §7. Chapter 9: §1 and §2.

Exercises from Folland: Chapter 8: 26, 44.

Problem 1: For $f \in L^2([0,\infty))$ define the function Tf on $[0,\infty)$ by

$$Tf(x) = \int_x^\infty \frac{f(t)}{t+1}\, dt.$$

Show that T is a one-to-one bounded linear map from $L^2([0,\infty))$ to $C_0([0,\infty))$ (the space of continuous functions on $[0,\infty)$ vanishing at infinity, with the supremum norm), and that $\|T\| = 1$.

Problem 2: Suppose that $\psi \in \mathcal{S}(\mathbb{R})$ ($\mathcal{S}(\mathbb{R})$ denotes the Schwartz class) satisfies $\int |\psi(x)|^2\, dx = 1$. Show that

$$\left(\int_{\mathbb{R}} |x|^2 |\psi(x)|^2\, dx\right)\left(\int_{\mathbb{R}} |\zeta|^2 |\hat{\psi}(\zeta)|^2\, d\zeta\right) \geq \frac{1}{16\pi^2}.$$

Problem 3: Let $1 \leq p < \infty$ and fix $f \in L^p(\mathbb{R})$ such that $f(x) = 0$ for $|x| \geq 1$. For $g \in L^q(\mathbb{R})$ where $\frac{1}{p} + \frac{1}{q} = 1$, let $f * g$ denote the convolution of f and g. Let $\mathcal{F}$ be the following family of functions

$$\mathcal{F} = \left\{ f * g;\ g \in L^q(\mathbb{R}), \|g\|_q \leq 1, \text{ and } g(x) = 0 \text{ for } |x| \geq 1 \right\}.$$

1. Show that $\mathcal{F} \subset C([-2,2])$, the space of continuous functions on the interval $[-2,2]$.
2. Show that the set $\mathcal{F}$ is precompact in $C([-2,2])$ (with the uniform norm).