Math 526

Homework due 05/28/2014

Reading from Folland 2nd edition: Chapter 8: §7. Chapter 9: §1.

Exercises from Folland: Chapter 8: 6

Problem 1 (Prelim 2004): Let $f \in L^1(\mathbb{R}^2)$ be a real-valued integrable function with respect to the usual 2-dimensional Lebesgue measure m. For $(x, y) \in \mathbb{R}^2$ let $D(x, y)$ denote the closed disk of radius 1 centered at (x, y). Define a function g by

$$g(x, y) = \int_{D(x,y)} f \, dm.$$

- Prove that g is continuous on $\mathbb{R}^2$.
- Show that $g(x, y) \to 0$ as $x^2 + y^2 \to \infty$.
- Prove that there exists at least one point (x, y) for which

$$|g(x, y)| \leq \frac{1}{x^2 + y^2 + 100}.$$

Problem 2 (Prelim 2004): Let f be a fixed but arbitrary Lebesgue measurable function on $\mathbb{R}$. For $p \in \mathbb{R}$ define

$$F(p) = \int_{\mathbb{R}} |f(t)|^p \, dt,$$

where we use the convention $0^0 = 0$. Show that

$$I = \{p : F(p) < \infty\}$$

is a connected subset of $\mathbb{R}$ and that $\log F$ is a convex function on this set.

Problem 3: For $x \in \mathbb{R}^n_+ = \{(\bar{x}, x_n) : \bar{x} \in \mathbb{R}^{n-1},\ x_n > 0\}$ and $y \in \partial\mathbb{R}^n_+$ define

$$K(x,y) = \frac{2x_n}{n\omega_n} \cdot \frac{1}{|x-y|^n}.$$

Here ω_n is the Lebesque measure of the unit ball in $\mathbb{R}^n$. Recall that $n\omega_n$ is the surface of the unit sphere in $\mathbb{R}^n$. Assume that $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$. For $x \in \mathbb{R}^n_+$ define

$$u(x) = \frac{2x_n}{n\omega_n} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n} dy$$

1. Show that $\int_{\partial\mathbb{R}^n_+} K(x,y)\, dy = 1$
2. Show that $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$,
3. $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = 0$ in $\mathbb{R}^n_+$,
4. $\lim\limits_{x \to x_0, x \in \mathbb{R}^n_+} u(x) = g(x_0)$ for each $x_0 \in \partial\mathbb{R}^n_+$.