Math 526

Homework due 05/07/2014

Stein & Shakarchi: Chapter 4: section 6.

Note: This homework assignment has 2 pages.

Problem 1 (Prelim 2012): Let $\{f_n\}_n$ be a sequence in $L^p(\mathbb{R})$ for some $1 < p < \infty$. Let $f \in L^p(\mathbb{R})$. Shoe that the following two assertions are equivalent:

1. $\sup_n \|f_n\| < \infty$ and $\lim_{n\to\infty} \int_0^x f_n(t)\,dt = \int_0^x f(t)\,dt$ for every $x \in \mathbb{R}$.
2. $\{f_n\}$ converges weakly to f in $L^p(\mathbb{R})$.

Problem 2 (Prelim 2011): Suppose $f \in L^1([0,1])$. Show that if

$$\int_0^1 f(x)(\sin x)^n\,dx = 0, \qquad \text{for all } n = 1, 2, \cdots$$

then $f = 0$ a.e.

Problem 3 (Prelim 2010): Let $C^1[0,1] = \{f \text{ and } f' \in C[0,1]\}$ where we take the right derivative of f at zero and the left derivative at one in defining f'. Equip $C^1[0,1]$ with the inner product

$$\langle f, g\rangle_1 = \int_0^1 f(t)g(t)dt + \int_0^1 f'(t)g'(t)dt.$$

This inner product, in a natural way, induces a norm and a metric on $C^1[0,1]$. Prove that any Cauchy sequence from $C^1[0,1]$ in the above metric converges to a continuous function. In other words, the completion of $C^1[0,1]$ in this metric is a subset of $C[0,1]$.

Problem 4 (Prelim 2008): Let (X,d) be a metric space. A map $f : X \to X$ is said to be an isometry of (X,d) if $d(f(x), f(y)) = d(x,y)$ for all $x,\ y \in X$. Show that if (X,d) is a compact metric space and $f : X \to X$ is an isometry, then f is surjective.

Problem 5 (Prelim 2006):

1. Let X be a topological space, let O be an open subset of X. Prove that the points of discontinuity of the characteristic function of O, χ_O, is a nowhere dense set of X. Recall that $A \subset X$ is nowhere dense if $\operatorname{int} \overline{\mathrm{A}} = \emptyset$.

2. Let X be a complete metric space, and $\{U_i\}_{i=1}^{\infty}$ any countable collection of open sets. Show that there exists $x \in X$ such that χ_{U_i} is continuous at x for each i.