Math 526

Homework due 04/23/2014

Reading from Folland, 2nd edition: Chapter 6: section 3.
Chapter 5: section 4 (partially).

Problems from Folland: Chapter 5: problem 32 and 35.

Problem 1: Suppose that the Banach space $\mathcal{X}$ is uniformly convex. That is, suppose that for every $\epsilon > 0$ there exists $\delta \in (0, 1)$ so that if

$$\|x\| = \|y\| = 1 \text{ and } \|x - y\| \geq \epsilon \text{ then } \|\frac{x+y}{2}\| < 1 - \delta.$$

Let f be a linear functional on $\mathcal{X}$ with norm 1. Prove that there is a unique point $x \in \mathcal{X}$ such that $\|x\| = 1$ and $f(x) = 1$.

Problem 2 (Reals 2011): Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ a one-to-one bounded linear map whose range $T(\mathcal{T})$ is closed in $\mathcal{Y}$. Show that for each bounded linear functional ϕ on $\mathcal{X}$ there is a bounded linear functional ψ on $\mathcal{Y}$ such that $\phi = \psi \circ T$, and there is a constant C (independent of ϕ) such that ψ can be chosen to satisfy $\|\psi\| \leq C\|\phi\|$.

Problem 3 (Reals 2006): 1. State the Open Mapping Theorem and the Closed Graph Theorem for Banach spaces. Derive the Open Mapping Theorem from the Closed Graph Theorem