Math 526

Homework due 04/09/2014

Reading from Folland, 2nd edition: Chapter 6: section 2 and 3.
Chapter 5: sections 1, 2 and 3.

Problems from Folland: Chapter 6: problem 20.
Chapter 5: problems 12 and 17.

Problem: Let $(X, \mathcal{M}, \mu)$ be a complete σ-finite measure space.

1. Show that for $p \geq 2$ and $x \in [0, 1]$

$$\left(\frac{1+x}{2}\right)^p + \left(\frac{1-x}{2}\right)^p \leq \frac{1}{2}(1 + x^p)$$

2. Prove that for $2 \leq p < \infty$, if $f,\ g \in L^p(\mu)$ then

$$\left\|\frac{f+g}{2}\right\|_p^p + \left\|\frac{f-g}{2}\right\|_p^p \leq \frac{1}{2}\left(\|f\|_p^p + \|g\|_p^p\right)$$

3. Show that for $1 < p < 2$ and $x \in [0, 1]$

$$(1+x)^q + (1-x)^q \leq 2(1 + x^p)^{\frac{q}{p}},$$

   where $\frac{1}{p} + \frac{1}{q} = 1$.

4. Prove that for $1 < p \leq 2$, if $f,\ g \in L^p(\mu)$ then

$$\left\|\frac{f+g}{2}\right\|_p^q + \left\|\frac{f-g}{2}\right\|_p^q \leq \left(\frac{\|f\|_p^p + \|g\|_p^p}{2}\right)^{q/p}$$

   where $\frac{1}{p} + \frac{1}{q} = 1$.

5. The inequalities above are called Clarkson inequalities. They imply that L^p is **uniformly convex**. This means that given $\epsilon > 0$ there is $\delta = \delta(\epsilon) \in (0, 1)$ with $\delta(\epsilon) \to 0$ as $\epsilon \to 0$ s so that whenever

$$\|f\|_p = \|g\|_p = 1 \quad \text{then} \quad \|f - g\|_p \geq \epsilon \text{ implies that } \left\| \frac{f+g}{2} \right\|_p \leq 1 - \delta.$$

6. Using the result above prove that for $1 < p < \infty$ the following statement holds. If $\{f_n\}_n \in L^p$ converges weakly to f and $\|f_n\|_p \to \|f\|_p$ then $\{f_n\}_n$ converges strongly to f, that is $\|f_n - f\|_p \to 0$ as $n \to \infty$.