Math 525

Homework due 03/05/2014

Reading from Folland: Chapter 4: sections 5, 6 and 7.

Problem 1 (1993): Let X denote the space $C([0,1])$ with the uniform norm. Show that X has no non-empty open subset with compact closure.

Problem 2: Let $(X, \mathcal{T})$ be a compact Hausdorff space, and let $E \subset X$. The relative topology on E is $\mathcal{T}_E = \{U \cap E : U \in \mathcal{T}\}$. Prove that:

1. if E is open, then E is locally compact in the relative topology. That is show that for every point $x \in E$ there exists an open set $U \in \mathcal{T}_E$ such that $x \in U$ and $\overline{U}$ is compact in $\mathcal{T}_E$.
2. if E is dense and locally compact in its relative topology, then E is open.
3. E is locally compact in the relative topology if and only if E is relatively open in $\overline{E}$.

Problem 3:

1. Let X and Y be topological spaces, let X be compact and $\pi_2 : X \times Y \to Y$ denote the projection onto Y. Let $A \subset X \times Y$ be a closed subset. Show that $\pi_2(A)$ is a closed subset of Y.
2. Let X and Y be topological spaces, and assume that Y is Hausdorff. Prove that if $f : X \to Y$ is continuous then graph f is a closed subset of $X \times Y$. Here

$$\text{graph}\, f = \{(x, f(x)) : x \in X\}\,.$$

   Show by an example that the converse is false.

Problem 4 (2010): Let $(X, \mathcal{M})$ be a measurable space on which two probability measures μ and ν exist (i.e. $\mu(X) = \nu(X) = 1$). Define the total variation metric between the two measures as

$$\|\mu - \nu\| = \sup_{A \in \mathcal{M}} |\mu(A) - \nu(A)|$$

Suppose that λ is a positive σ-finite measure on $\mathcal{M}$ such that μ and ν are dominated by λ (i.e $\mu \ll \lambda$ and $\nu \ll \lambda$). Prove that

$$\|\mu - \nu\| = \frac{1}{2} \int_X \left| \frac{d\mu}{d\lambda} - \frac{d\nu}{d\lambda} \right| d\lambda.$$

Here $d\mu/d\lambda$ and $d\nu/d\lambda$ refer to Radon-Nikodym derivatives of μ with respect to λ and ν with respect to λ respectively.

Problem 5 (2006): Let

$$\mathcal{H} = \left\{ u : \mathbb{R}^n \to \mathbb{R} : u \text{ continuous and } \forall x \in \mathbb{R}^n, \forall r > 0 u(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} u(y)\, dy \right\}$$

Show that if $\{u_n\}_{n=1}^{\infty} \subset \mathcal{H}$ is uniformly bounded on compact subsets of $\mathbb{R}^n$, there exists a subsequence that converges uniformly to a function $u \in \mathcal{H}$ on compact subsets of $\mathbb{R}^n$.