Math 525

Homework due 02/19/2014

Reading from Folland: Chapter 4: sections 1, 2, 4.

Problem 1: Let μ be a Borel measure in $\mathbb{R}^d$. Suppose there exists a constant $C > 1$ such that such for each $x \in \mathbb{R}^d$ and $r > 0$

$$C^{-1} r^d \leq \mu(B(x, r)) \leq C r^d.$$

- Prove that μ is absolutely continuous with respect to the Lebesgue measure m in $\mathbb{R}^d$.
- Let f denote the Radon-Nikodym derivative of μ with respect to m, i.e. $d\,\mu = f\, d\, m$. Show that for m-a.e. $x \in \mathbb{R}^d$

$$\frac{C^{-1}}{\omega_d} \leq f(x) \leq \frac{C}{\omega_d},$$

where ω_d denotes the Lebesgue measure of the unit ball in $\mathbb{R}^d$.

Problem 2: Let $f \in BV[a, b]$ where $-\infty < a < b < \infty$. Prove that

1. $f = g + h$ where g is absolutely continuous and $[a, b]$ and $h' = 0$ a.e. on $[a, b]$, and
2. the decomposition above is unique up to additive constants.

Problem 3: Let (X, d) be a metric space. Prove that if A and B are closed disjoint subsets of X there exist open disjoint sets U and V such that $A \subset U$ and $B \subset V$.

Problem 4: Let X be a topological space. Let $U \subset X$ be an open set. Prove that if A is dense in X then $\overline{U} = \overline{U \cap A}$.