Review problems

March 10, 2014

Problem 1: Let f be a real valued function on $[0,1]$. Prove that f is absolutely continuous on $[0,1]$ if and only if there exists a sequence $\{f_n\}_{n\geq 1}$ of Lipschitz functions on $[0,1]$ such that $T_0^1(f-f_n)\to 0$ as $n\to\infty$.

Recall that f is Lipschitz on $[0,1]$ if there exists $M>0$ such that

$$|f(x)-f(y)|\leq M|x-y| \quad \text{whenever} \quad x,\, y\in[0,1].$$

Problem 2 (2004): Let $C^1(\mathbb{R})$ denote the space of continuously differentiable real-valued functions on $\mathbb{R}$. Suppose that $f_n\in C^1(\mathbb{R})$ for $n\geq 1$ and $f_n(x)=0$ for all $|x|>1$ and all $n\in\mathbb{N}$. Assume that there is a $p>1$ such that $\|f_n'\|_p\leq 1$ for all $n\in\mathbb{N}$. Prove that there is a subsequence of the f_n's which converges uniformly on $\mathbb{R}$.

Problem 3: Does there exist a function $f\in C[0,1]$ such that

$$\int_0^1 xf(x)\,dx=1$$

and

$$\int_0^1 x^n f(x)\,dx=0 \quad \text{for each} \quad n=0,2,3,\cdots?$$

Justify your answer.

Problem 4 (2000): Let $(X,\mathcal{M},\mu)$ be a measure space. Let $\{f_n\}_{n\geq 1}$ be a sequence of non-negative measurable functions such that

$$\int_X f_n\,d\mu\leq 1.$$

Prove that

$$\limsup_{n\to\infty}(f_n(x))^{1/n}\leq 1 \quad \mu\ a.e.\ x\in X.$$

Problem 5: Let (X, d, μ) be a measure metric space. Assume that μ is doubling and locally finite, that is there exists a constant $C > 0$ such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) \text{ for all } x \in X \text{ and } r > 0.$$

Let $A \subset X$ and let $\mathcal{F}$ be a fine cover of A by non-degenerate closed balls. Prove that there exists a countable disjoint subcollection $\{B_i\}_i \subset \mathcal{F}$ such that

$$\mu\left(A \setminus \bigcup_{B \in \mathcal{G}} B\right) = 0.$$