Covering Theorems

We present Vitali's Covering Theorem, which plays a crucial role in the study of differentiation properties of functions and measures in Euclidean spaces.

Notation: If B is a closed ball in $\mathbb{R}^n$ of center x and radius r then $\hat{B}$ denotes the closed ball of center x and radius $5r$.

Definitions:

1. A collection $\mathcal{F}$ of closed balls in $\mathbb{R}^n$ is a **cover** of a set $A \subset \mathbb{R}^n$ if

$$A \subset \bigcup_{B \in \mathcal{F}} B$$

2. $\mathcal{F}$ is a **fine cover** of A if, in addition,

$$\inf \{\operatorname{diam} B | x \in B, B \in \mathcal{F}\} = 0$$

for each $x \in A$.

Vitali's Covering Theorem:

Let $\mathcal{F}$ be any collection of non-degenerate closed balls in $\mathbb{R}^n$ with

$$\sup \{\operatorname{diam} B | B \in \mathcal{F}\} < \infty.$$

Then there is a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{G}} \hat{B}$$

Corollary 1: Assume that $\mathcal{F}$ is a fine cover of A by closed balls and

$$\sup\{\operatorname{diam} B | B \in \mathcal{F}\} < \infty.$$

Then there is a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that for each finite subset $\{B_1, B_2, \cdots, B_m\} \subset \mathcal{F}$, we have

$$A \setminus \bigcup_{k=1}^{m} B_k \subset \bigcup_{B \in \mathcal{G} \setminus \{B_1, B_2, \cdots, B_m\}} \hat{B}$$

Corollary 2: Let $U \subset \mathbb{R}^n$ be an open set. Let $\delta > 0$. There exists a countable collection $\mathcal{G}$ of disjoint closed balls in U such that $\operatorname{diam} B \leq \delta$ for all $B \in \mathcal{G}$ and

$$m\left(U \setminus \bigcup_{B \in \mathcal{G}} B\right) = 0.$$

Corollary 3: Let $E \subset \mathbb{R}^n$ be a measurable set such that $\mathcal{L}^n(E) < \infty$. Let $\mathcal{F}$ be a fine cover of E. Given $\epsilon > 0$ there is a finite disjoint subcollection $\{B_1, B_2, \cdots, B_k\}$ of $\mathcal{F}$ such that

$$m\left(E \setminus \bigcup_{j=1}^{k} B_j\right) < \epsilon.$$