Real Analysis - Prelim problems

Winter 2010

1. (2005) Given $A \subset \mathbb{R}$ define a sequence $\{A_k\}_{k\geq 1}$ of sets inductively by $A_1 = A$, and for $k \geq 1$, $A_{2k} = \bar{A}_{2k-1}$ and $A_{2k+1} = \operatorname{int} A_{2k}$.

   - Find $A \subset \mathbb{R}$ such that the family $\{A_k\}_{k\geq 1}$ contains 4 distinct sets.
   - Prove that for any $A \subset \mathbb{R}$, the family $\{A_k\}_{k\geq 1}$ contains at most four distinct sets.

2. (2005) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, and there exists $k > 0$ such that for every $y \in \mathbb{R}$ there are at most k distinct points $x \in \mathbb{R}$ with $f(x) = y$. Prove that the derivative of f exists for a.e. $x \in \mathbb{R}$.

3. (1997) Let $\{X_\alpha\}_{\alpha\in A}$ be a family of Hausdorff spaces and let X be their product $\Pi_{\alpha\in A} X_\alpha$, endowed with the product topology.

   - Show that X is Hausdorff.
   - Assume that infinitely many of the spaces X_α, $\alpha \in A$ are non-compact. Show that every compact subset of X is nowhere dense.

4. (2006) Suppose that $(X, \mathcal{M}, \mu)$ is a finite measure space, and suppose that λ is a **finitely** additive, nonnegative, real valued set function on $\mathcal{M}$ for which:

$$\forall\ \epsilon > 0, \exists\ \delta > 0,\ \forall E \in \mathcal{M} :\ \mu(E) < \delta \Rightarrow \lambda(E) < \epsilon.$$

   Prove that there exists $g \in L^1(\mu)$ such that for all $E \in \mathcal{M}$:

$$\lambda(E) = \int_E g\, d\mu.$$

5. (1991) Let $\mathcal{H}$ be a separable, infinite dimensional Hilbert space with the norm $\|\cdot\|$. Let x_n and x be such that $\|x_n\| \to \|x\|$ as $n \to \infty$. Moreover suppose that for every continuous linear functional φ on $\mathcal{H}$, $\varphi(x_n) \to \varphi(x)$ as $n \to \infty$. Show that $\|x_n - x\| \to 0$ as $n \to \infty$.

6. Let f be a real valued function on $[0,1]$. Prove that f is absolutely continuous on $[0,1]$ if and only if there exists a sequence $\{f_n\}_{n\geq 1}$ of Lipschitz functions on $[0,1]$ such that $V_0^1(f-f_n)\to 0$ as $n\to\infty$.

   Recall that f is Lipschitz on $[0,1]$ if there exists $M>0$ such that

$$|f(x)-f(y)|\leq M|x-y| \quad \text{whenever} \quad x,\, y\in[0,1].$$

7. Let μ be a finite signed regular Borel measure on $[0,1]$. Suppose that

$$\int_0^1 e^{-nx}\,d\mu(x)=0, \quad n=0,1,2,\cdots.$$

   Prove that $\mu=0$.

   Recall that a signed Borel measure μ is regular if both μ^+ and μ^- are Borel regular measures.