Math 525

Homework due 01/13/2010

Reading from Stein & Shakarchi: Chapter 3: sections 1 and 2.

Problem 1 Let μ be a finite Borel measure on $\mathbb{R}^n$. Let

$$S_\infty = \{x \in \mathbb{R}^n : \limsup_{\rho \to 0} \frac{\mu(B(x,\rho))}{\rho} = \infty\}.$$

Prove that for each $\epsilon > 0$ there exists a covering of S_∞ by balls $B(x_j, \rho_j)$ such that

$$\sum_j \rho_j < \epsilon.$$

Remark: This shows that S_∞ has 1-dimensional Hausdorff measure 0.

Problem 2: Let μ be a σ-finite Borel measure in $\mathbb{R}^n$ satisfying $\mu(B(0,1)) < \infty$ and

$$(*) \qquad \mu(B(x,2r)) \leq C\mu(B(x,r)) \quad \text{for every } x \in \mathbb{R}^n \text{ and } r > 0,$$

for some fixed constant C. A measure satisfying $(*)$ is called a doubling measure. Prove that given an open set $U \subset \mathbb{R}^n$ and $\delta > 0$, there exists a countable collection $\mathcal{G}$ of disjoint closed balls in U such that $\operatorname{diam} B \leq \delta$ for all $B \in \mathcal{G}$ and

$$\mu\left(U \setminus \bigcup_{B \in \mathcal{G}} B\right) = 0.$$

Exercises from Stein & Shakarchi: Chapter 3: 4 and 5.