Definitions

- A subset $\mathcal{A}$ of $C(X, \mathbb{R})$ or $C(X, \mathbb{C})$ is said to **separate points** if for every $x,\ y \in X$ with $x \neq y$ there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.
- $\mathcal{A}$ is called and **algebra** if it is a real (resp. complex) vector subspace of $C(X, \mathbb{R})$ (resp. $C(X, \mathbb{C})$) such that $fg \in \mathcal{A}$ whenver $f,\ g \in \mathcal{A}$.
- If $\mathcal{A} \subset C(X, \mathbb{R})$, $\mathcal{A}$ is called a **lattice** if $f \vee g = \max\{f, g\}$ and $f \wedge g = \min\{f, g\}$ are in $\mathcal{A}$ whenver $f,\ g \in \mathcal{A}$.

Note: If $\mathcal{A}$ is an algebra or a lattice, so is $\overline{\mathcal{A}}$.

Stone-Weierstrass Theorem

Theorem: Let X be a compact Hausdorff space. If $\mathcal{A}$ is a closed subalgebra of $C(X, \mathbb{R})$ that separates points, then either:

- $\mathcal{A} = C(X, \mathbb{R})$ or
- $\mathcal{A} = \{f \in C(X, \mathbb{R}) : \ f(x_0) = 0\}$ for some $x_0 \in X$.

The first alternative holds if and only if $\mathcal{A}$ contains the constant functions.

Corollary: Suppose $\mathcal{B}$ is a subalgebra of $C(X, \mathbb{R})$ that separates points and contains the constants then $\mathcal{B}$ is dense in $C(X, \mathbb{R})$.

Corollary: Every continuous function on a compact set X of $\mathbb{R}^n$ can be uniformly approximated on X by a plynomial (in the coordinates).

Lemma 1: Consider $\mathbb{R}^2$ as an algebra under coordinatewise addition and multiplication. Then the only subalgebras of $\mathbb{R}^2$ are $\mathbb{R}^2$, $\{(0,0)\}$ and the linear spans of $(1,0)$, $(0,1)$ or $(1,1)$.

Lemma 2: For any $\epsilon > 0$ there is a polynomial P on $\mathbb{R}$ such that $P(0) = 0$ and $||x| - P(x)| < \epsilon$ for $x \in [-1,1]$.