Continuous functions on LCH spaces

- **Proposition:** If X is a LCH space, $C_0(X) = \overline{C_c(X)}$ in the uniform metric.
- **Lemma:** If X is a LCH space and $E \subset X$, then E is closed if and only if $E \cap K$ is closed for every compact set $K \subset X$.
- **Proposition:** If X is a LCH space, $C(X)$ is a closed subset of $\mathbb{R}^X$ in the topology of uniform convergence on compact sets.

σ-compactness

- X is σ-compact if $X = \cup_{i=1}^{\infty} K_i$ and each K_i is compact.

Proposition: If X is a σ-compact, LCH space there is a sequence $\{U_n\}_n$ of precompact open sets such that

$$X = \cup_n U_n \ \& \ \ \overline{U_n} \subset U_{n+1}.$$

Moreover if $f \in \mathbb{R}^X$ the sets

$$\left\{ g \in \mathbb{R}^X : \sup_{x \in \overline{U_n}} |g(x) - f(x)| < \frac{1}{m} \right\} \qquad n, m \in \mathbb{N}$$

form a neighborhood base for f in the topology of uniform convergence on compact sets. This topology is 1st countable, and $f_n \to f$ uniformly on compact sets if and only if $f_n \to f$ on each $\overline{U_n}$.

Partitions of unity

If X is a topological space and $E \subset X$ a partition of unity on E is a collection $\{h_\alpha\}_\alpha$ with $h_\alpha \in C(X, [0,1])$ such that

- For each $x \in X$ there is a neighborhood containing x where only finitely many h_α's are non-zero.
- $\sum_\alpha h_\alpha(x) = 1$ for $x \in E$.

A partition of unity $\{h_\alpha\}_\alpha$ is subordinate to an open cover $\mathcal{U}$ of E if for each α there exists $U \in \mathcal{U}$ such that $\operatorname{supp} h_\alpha \subset U$.

Proposition: Let X be a LCH space, $K \subset X$ compact and $\{U_j\}_{j=1}^n$ an open cover of K. There is a partition of unity on K subordinate to $\{U_j\}_{j=1}^n$ consisting of compactly supported functions.

Let X be a topological space and $\mathcal{F} \subset C(X)$.

- $\mathcal{F}$ is **equicontinuous at** $x \in X$ if for every $\epsilon > 0$ there is an open set U containing x such that $|f(x) - f(y)| < \epsilon$ for all $y \in U$ and $f \in \mathcal{F}$.
- $\mathcal{F}$ is **equicontinuous** if it is equicontinuous at each point $x \in X$.
- $\mathcal{F}$ is **pointwise bounded** if $\{f(x) : f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{R}$ or $\mathbb{C}$ for each $x \in X$.

Arzelà-Ascoli Theorem I

Let X be a compact Hausdorff space. If $\mathcal{F}$ is an equicontinuous, pointwise bounded subset on $C(X)$, then $\mathcal{F}$ is totally bounded in the uniform metric, and the closure of $\mathcal{F}$ in $C(X)$ is compact.

Arzelà-Ascoli Theorem II

Let X be a σ-compact LCH space. If $\{f_n\}$ is an equicontinuous, pointwise bounded sequence in $C(X)$, there exists $f \in C(X)$ and a subsequence of $\{f_n\}$ that converges to f uniformly on compact sets.