Compact Spaces -Definitions

- A topological space X is compact if whenever $\{U_\alpha\}_\alpha$ is an open cover of X, i.e. $X = \cup_\alpha U_\alpha$ there exists a finite subset $\{U_i\}_{i=1}^m \subset \{U_\alpha\}_\alpha$ such that $X = \cup_{i=1}^m U_i$.
- $Y \subset X$ is compact if whenever $\{U_\alpha\}_\alpha$ is a collection of open sets in X with $Y \subset \cup_\alpha U_\alpha$ there exists a finite subset $\{U_i\}_{i=1}^m \subset \{U_\alpha\}_\alpha$ such that $Y \subset \cup_{i=1}^m U_i$.
- $Y \subset X$ is precompact if $\overline{Y}$ is compact.
- A family $\{F_\alpha\}_{\alpha \in A} \subset \mathcal{P}(X)$ has the finite intersection property if $\cap_{\alpha \in B} F_\alpha \neq \emptyset$ for all finite $B \subset A$.

Properties

Proposition: A topological space is compact if and only if for every family of closed sets with the finite intersection property $\{F_\alpha\}_{\alpha \in A}$ then $\cap_{\alpha \in A} F_\alpha \neq \emptyset$.

Proposition: A closed subset of a compact space is compact.

Finite intersection property I

Lemma: Let $\mathcal{A}$ be a collection of subsets of a set X, and suppose that $\mathcal{A}$ has the finite intersection property. Then there is a collection $\mathcal{B}$ such that $\mathcal{A} \subset \mathcal{B}$, $\mathcal{B}$ has the finite intersection property and it is maximal with respect to this property: that is, no collection properly containing $\mathcal{B}$ has the finite intersection property.

Finite intersection property II

Lemma: Let $\mathcal{B}$ be a collection of subsets of X that is maximal with respect to the finite intersection property. Then each intersection of a finite number of sets in $\mathcal{B}$ is again in $\mathcal{B}$, and each set that meets every set in $\mathcal{B}$ is itself in $\mathcal{B}$.

Product of compact spaces

Theorem (Tychonoff): Let $\{X_\alpha\}_\alpha$ be a family of compact topological spaces. Then the product space $\Pi_\alpha X_\alpha$ is compact with the product topology.