Properties of the Cartesian Product

Proposition: If X_α is a Hausdorff space for each α then $X = \Pi_\alpha X_\alpha$ is also a Hausdorff space with the product topology.

Proposition: If $\{X_\alpha\}_\alpha$ and Y are topological spaces, and $X = \Pi_\alpha X_\alpha$ then $f : Y \to X$ is continuous is and only if $\pi_\alpha \circ f : Y \to X_\alpha$ is continuous for each α.

If for each $\alpha \in A$, $X_\alpha = X$ then

$$X = \Pi_{\alpha \in A} X_\alpha = X^A = \{f : A \to X\}$$

Proposition: If X is a topological space, $A \neq \emptyset$ and $\{f_n\}$ is a sequence in X^A then $f_n \to f$ in the product topology if and only if $f_n \to f$ pointwise (in A).

Continuous real-valued functions

Let $(X, \mathcal{T})$ be a topological space.

- $B(X, \mathbb{R}) = B(X) = \{f : X \to \mathbb{R} : \ f \ \text{ bounded}\}$
- $C(X, \mathbb{R}) = C(X) = \{f : X \to \mathbb{R} : \ f \ \text{ continuous}\}$
- $BC(X) = B(X) \cap C(X)$

If $f \in B(X)$ we define the **uniform norm**

$$\begin{aligned} \|f\|_u &= \sup\{|f(x)| : \ x \in X\}. \\ \rho(f, g) &= \|f - g\|_u \quad \text{is a metric on} \quad B(X). \end{aligned}$$

Convergence in this metric is called **uniform convergence**.

Properties

Remark: $B(X)$ with the uniform metric is a complete metric space.

Proposition: If X is a topological space, $BC(X)$ is a closed subspace of $B(X)$ with the uniform metric. In particular $BC(X)$ is complete.

Question: How big is $C(X)$? Consider $(\mathbb{N}, \mathcal{T})$ where $\mathcal{T}$ denotes the co-finite topology.

If X is T_4, $C(X)$ is big.

Lemma: Suppose that X is T_4, and $A,\ B$ are closed disjoint sets on X. Let

$$\Delta = \{k2^{-n}, n \in \mathbb{N} \ \ 0 < k < 2^n\}$$

be the dyadic rationals in (0,1).

There is a family $\{U_r : r \in \Delta\}$ of open sets in X such that

$$A \subset U_r \subset B^c, \quad \forall\ r \in \Delta \quad \text{and} \quad \overline{U}_r \subset U_s \quad \text{of} \quad r < s.$$

Separation of closed sets by continuous functions on T_4 spaces

Urysohn Lemma: Let X be a T_4 space, and let A, B be closed disjoint sets on X. There exits $f \in C(X, [0,1])$ such that $f = 0$ on A and $f = 1$ on B.

Extension of continuous functions defined on closed sets on T_4 spaces

Tietze Extension Theorem: Let X be a T_4 space, and let A be a closed set on X. Let $f \in C(A, [0,1])$. There exits $F \in C(X, [0,1])$ such that $F_{|A} = f$.

Corollary: Let X be a T_4 space, and let A be a closed set on X. Let $f \in C(A)$. There exits $F \in C(X)$ such that $F_{|A} = f$.