Pointwise convergence

Definition: A sequence $\{x_j\}_j$ is a topological space $(X, \mathcal{T})$ converges to $x \in X$ (i.e. $x_j \to x$) if and only if for all $U \in \mathcal{T}$ with $x \in U$ there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$, $x_j \in U$.

Proposition: If X is first countable and $A \subset X$ then $x \in \overline{A}$ if and only if there exits a sequence $\{x_j\}_j \subset A$ such that $x_j \to x$.

Example

Let $(\mathbb{N}, \mathcal{T})$ be such that $\mathcal{T} = \{U \subset \mathbb{N} : \ U = \emptyset \ \text{ or } \ U^c \text{ finite}\}$.

- $(\mathbb{N}, \mathcal{T})$ first countable.
- What sequences converge?

Separation axioms

Let $(X, \mathcal{T})$ a topological space.

- T_0: If $x \neq y$ $\exists U \in \mathcal{T}$ such that $x \in U$ and $y \notin U$ or $\exists V \in \mathcal{T}$ such that $x \notin V$ and $y \in V$.
- T_1: If $x \neq y$ $\exists U \in \mathcal{T}$ such that $x \in U$ and $y \notin U$ and $\exists V \in \mathcal{T}$ such that $x \notin V$ and $y \in V$.
- T_2: If $x \neq y$ $\exists U,\ V \in \mathcal{T}$ such that $U \cap V = \emptyset$, $x \in U$ and $y \in V$.
- T_3: X is T_1 and for all $A \subset X$ closed with $x \in A^c$, $\exists U,\ V \in \mathcal{T}$ such that $U \cap V = \emptyset$, $x \in U$ and $A \subset V$.
- T_4: X is T_1 and for all $A,\ B \subset X$ closed with $A \cap B = \emptyset$ $\exists U,\ V \in \mathcal{T}$ such that $U \cap V = \emptyset$, $A \subset U$ and $B \subset V$.

Separation terminology

- If $(X, \mathcal{T})$ is T_2, X is a Hausdorff space.
- If $(X, \mathcal{T})$ is T_3, X is a regular space.
- If $(X, \mathcal{T})$ is T_4, X is a normal space.

Proposition: $(X, \mathcal{T})$ is T_1 if and only if $\{x\}$ is closed for every $x \in X$.

Continuous maps

- Let X and Y be topological spaces. Let $f : X \to Y$, f is said to be **continuous** if for every open set $V \subset Y$ $f^{-1}(V)$ is open in X.

$$C(X, Y) = \{f : X \to Y : \ f \ \text{ continuous } \}.$$

- Let X and Y be topological spaces. Let $f : X \to Y$, f is said to be **continuous** at $x \in X$ if for every open set $V \subset Y$ containing $f(x)$ there is an open set $U \subset X$ containing x such that $f(U) \subset V$.

Proposition: The map $f : X \to Y$ is continuous if and only if f is continuous at every $x \in X$.

Proposition: If the topology in Y is generated by $\mathcal{E}$, then the map $f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is open in X for every $V \in \mathcal{E}$.

Definitions:

- **Homeomorphism:** $f : X \to Y$ bijective and f and f^{-1} continuous. X and Y are **homeomorphic**.

- Let $X \neq \emptyset$ be a set, $\{Y_\alpha\}_\alpha$ a collection of topological spaces and $\{f_\alpha : X \to Y_\alpha\}_\alpha$ a family of maps from X into Y_α. There exists a unique weakest topology $\mathcal{T}$ on X that makes all f_α continuous. $\mathcal{T}$ is the **weak topology** generated by $\{f_\alpha\}$. $\mathcal{T}$ is generated by the sets of the form

$$\left\{ f_\alpha^{-1}(U_\alpha) : U_\alpha \text{ open in } Y_\alpha \right\}.$$

Cartesian Product

If $\{X_\alpha\}_\alpha$ is a family of topological spaces the **product topology** on $X = \Pi_\alpha X_\alpha$ is the weak topoloy generated by the projection maps

$$\pi_\alpha : X \to X_\alpha.$$

A base for the product topology is given by the sets of the form

$$\bigcap_{j=1}^{n} \pi_{\alpha_j}^{-1}(U_{\alpha_j}) \quad n \in \mathbb{N}, \quad U_{\alpha_j} \subset X_{\alpha_j} \text{ open for } 1 \leq j \leq n.$$

These sets can be written as $\Pi_\alpha U_\alpha$ where $U_\alpha = X_\alpha$ except for finitely many α's.