Definitions

Topology: Let X be a non-empty set. A **topology** on X is a family $\mathcal{T} \subset \mathcal{P}(X)$ such that:

- $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$
- If $O_1,\ O_2\ \in \mathcal{T}$ then $O_1 \cap O_2 \in \mathcal{T}$
- If $\{O_\alpha\}_\alpha \subset \mathcal{T}$ then $\cup_\alpha O_\alpha \in \mathcal{T}$.

$(x, \mathcal{T})$ is a **topological space**.

Examples

- If X is a metric space, the collection of all open sets with respect to the metric is a topology on X.
- If $(X, \mathcal{T})$ is a topological space, $Y \subset X$ then $\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}\}$ is a topology in Y, called the **relative topology** induced by $\mathcal{T}$.

Terminology

Let $(X, \mathcal{T})$ be a topological space.

- U is **open** if and only if $U \in \mathcal{T}$.
- F is **closed** if and only if F^c is open.
- If $Y \subset X$ the open (closed) subsets of Y in the relative topology are called **relatively** open (closed).

Remark: If $\{F_\alpha\}_\alpha$ is a collection of closed sets then $\cap_\alpha F_\alpha$ is also closed.

Definitions

- If $A \subset X$ the **interior** of A is the largest open set contained in A, i.e.

$$\operatorname{int} A \quad = \bigcup_{U \in \mathcal{T}, U \subset A} U.$$

- If $A \subset X$ the **closure** of A is the smallest closed set containing in A, i.e.

$$\overline{A} \quad = \bigcap_{F^c \in \mathcal{T}, A \subset F} F.$$

- $(\operatorname{int} A)^c = \overline{A^c}$
- $(\overline{A})^c = \operatorname{int}(A^c)$.

Definitions

Let $(X, \mathcal{T})$ be a topological space.

- **Boundary** of A: $\partial A = \overline{A} \backslash \text{int} A = \overline{A} \cap \overline{A^c}$.
- A is **dense** in X if $\overline{A} = X$.
- A is **nowhere dense** if $\text{int}\, \overline{A} = \emptyset$.
- x is an **accumulation point** of A if $A \cap U \backslash \{x\} \neq \emptyset$ for every open set U containing x.

Proposition: If $A \subset X$, let

$$\operatorname{acc} A = \{x \in X : x \text{ accumulation points of } A\}.$$

Then $\overline{A} = A \cup \operatorname{acc} A$, and A is closed if and only if $\operatorname{acc} A \subset A$.

Terminology

Let $(X, \mathcal{T})$ be a topological space.

- Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X such that $\mathcal{T}_1 \subset \mathcal{T}_2$: $\mathcal{T}_1$ is **weaker** (**coarser**) than $\mathcal{T}_2$, $\mathcal{T}_2$ is **stronger** (**finer**) than $\mathcal{T}_1$.
- If $\mathcal{E} \subset \mathcal{P}(X)$ there exists a unique weakest topology $\mathcal{T}(\mathcal{E})$, containing $\mathcal{E}$. $\mathcal{T}(\mathcal{E})$ the intersection of all topologies on X that contain $\mathcal{E}$, it is the topology generated by $\mathcal{E}$.
- For $x \in X$, $\mathcal{B}_x \subset \mathcal{T}$ is a base at x if $x \in B$ for all $B \in \mathcal{B}_x$, and for each $U \in \mathcal{T}$ with $x \in U$ there is $B \in \mathcal{B}_x$ such that $x \in B \subset U$.
- $\mathcal{B} \subset \mathcal{T}$ is a base for $\mathcal{T}$ if for all $U \in \mathcal{T}$, and for all $x \in U$ there exits $B \in \mathcal{B}$ such that $x \in B \subset U$.

Proposition: A collection $\mathcal{B} \subset \mathcal{P}(X)$ is a base for some topology on X if and only if:

1. For all $x \in X$ there is $B \in \mathcal{B}$ such that $x \in B$.
2. If $B_1,\ B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there is $B_3 \in \mathcal{B}$ such that

$$x \in B_3 \subset B_1 \cap B_2.$$

Terminology

- $(X, \mathcal{T})$ is **first countable** if there is a countable base at each point.
- $(X, \mathcal{T})$ is **second countable** if $\mathcal{T}$ has a countable base.
- $(X, \mathcal{T})$ is **separable** if X has a countable dense set.

Proposition: Every second countable space is separable.