Orthogonal Projection

Let $\mathcal{H}$ be a Hilbert space. If $\mathcal{S}$ is a closed subspace then

$$\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^{\perp},$$

i.e. for $f \in \mathcal{H}$

$$f = g + h \quad \text{with} \quad g \in \mathcal{S} \quad \text{and} \quad h \in \mathcal{S}^{\perp}.$$

The mapping $\mathcal{P}_{\mathcal{S}} : \mathcal{H} \to \mathcal{S}$ defined by $\mathcal{P}_{\mathcal{S}}(f) = g$ is the orthogonal projections onto $\mathcal{S}$.

Properties of $\mathcal{P}_{\mathcal{S}} : \mathcal{H} \to \mathcal{S}$

1. $\mathcal{P}_{\mathcal{S}}$ is linear
2. $\mathcal{P}_{\mathcal{S}}(f) = f$ iff $f \in \mathcal{S}$.
3. $\mathcal{P}_{\mathcal{S}}(f) = 0$ iff $f \in \mathcal{S}^{\perp}$.
4. $\|\mathcal{P}_{\mathcal{S}}(f)\| \leq \|f\|$ for all $f \in \mathcal{H}$.

Remark: Let $\{e_k\}$ be a collection of o.n. vectors in $\mathcal{H}$. The orthogonal projection $\mathcal{P}$ in the closure of $\operatorname{span}\langle e_k \rangle$ is given by

$$\mathcal{P}(f) = \sum_k (f, e_k) e_k.$$

Linear Transformations

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ bet Hilbert spaces. A mapping $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a **linear transformation** or **linear operator** if

$$T(\alpha f + \beta g) = \alpha T(f) + \beta T(g) \quad \alpha,\ \beta \text{ scalars} \quad f, g \in \mathcal{H}_1.$$

- A linear operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ is **bounded** if there exists $M > 0$ such that

$$\|T(f)\|_2 \leq M\|f\|_1.$$

- The **norm** of T

$$\begin{aligned} \|T\| &= \sup\{\|Tf\|_2 : \|f\|_1 = 1\} \\ &= \sup\{\frac{\|Tf\|_2}{\|f\|_1} : f \neq 0\} \\ &= \inf\{M : \|Tf\|_2 \leq M\|f\|_1 \text{ for all } f \in \mathcal{H}_1\}. \end{aligned}$$

Examples

- The **identity operator** $I : \mathcal{H} \to \mathcal{H}$, $I(f) = f$, $\|I\| = 1$.
- Unitary operators and projections have norm 1.

Lemma:

$$\|T\| = \sup\{|(Tf, g)| : \ f \in \mathcal{H}_1, \ g \in \mathcal{H}_2 \ \text{ with } \ \|f\| \leq 1, \ \|g\| \leq 1\}.$$

Continuous linear transformations

Definition: A linear operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ is **continuous** if $T(f_n) \to T(f)$ whenever $f_n \to f$.

Proposition: A linear operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ is continuous if and only if it is bounded.

Linear functionals

Definition: A linear functional $\ell = \mathcal{H} \to \mathbb{C}$ is a linear transformation.

Remark: Given $g \in \mathcal{H}$ the map $\ell(f) = (f, g)$ is a linear functional. By Cauchy-Schwarz

$$|(f, g)| \leq M\|f\| \quad \text{where} \quad M = \|g\|.$$

Riesz representation theorem

Theorem: Let ℓ be a continuous linear functional on a Hilbert space $\mathcal{H}$. There exists a unique $g \in \mathcal{H}$ such that

$$\ell(f) = (f, g) \quad \text{for all } f \in \mathcal{H}.$$

Adjoints

Proposition: Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear transformation. There exits a unique bounded linear transformation $T^* : \mathcal{H} \to \mathcal{H}$ such that:

- $(Tf, g) = (f, T^*g)$.
- $\|T\| = \|T^*\|$.
- $(T^*)^* = T$.

T^* is the **adjoint** of T.