CONSTRUCTION OF THE REAL NUMBERS

We present a brief sketch of the construction of $\mathbb{R}$ from $\mathbb{Q}$ using Dedekind cuts. This is the same approach used in Rudin's book *Principles of Mathematical Analysis* (see Appendix, Chapter 1 for the complete proof). The elements of $\mathbb{R}$ are some subsets of $\mathbb{Q}$ called cuts. On the collection of these subsets, i.e. on $\mathbb{R}$, we define an order, an addition, and a multiplication. We show that $\mathbb{R}$ endowed with this relation and these two operations is an ordered field. Each rational number can be identified with a specific cut, in such a way that $\mathbb{Q}$ can be viewed as a subfield of $\mathbb{R}$.

Step 1. A subset α of $\mathbb{Q}$ is said to be a cut if:

1. α is not empty, $\alpha \neq \mathbb{Q}$.
2. If $p \in \alpha$, $q \in \mathbb{Q}$, and $q < p$, then $q \in \alpha$.
3. If $p \in \alpha$, then $p < r$ for some $r \in \alpha$.

Remarks:

- 3 implies that α has no largest number.
- 2 implies that:
  - If $p \in \alpha$ and $q \notin \alpha$ then $p < q$.
  - If $r \notin \alpha$ and $r < s$ then $s \notin \alpha$.

Example: Let

$$\alpha = \{p \in \mathbb{Q} : p < 0\} \cup \{p \in \mathbb{Q} : p \geq 0 \text{ and } p^2 < 2\}.$$

Note that α is a cut. In fact:

1. $\alpha \subset \mathbb{Q}$, $1 \in \alpha$ thus $\alpha \neq$, and $2 \notin \alpha$ thus $\alpha \neq \mathbb{Q}$.
2. If $p \in \alpha$, $q \in \mathbb{Q}$, and $q < p$, then either $q \leq 0$ and so $q \in \alpha$, or $q > 0$ which implies $p > 0$. But since $p \in \alpha$, then $p^2 < 2$. Since $0 < q < p$ then $q^2 < p^2$. Therefore $q^2 < 2$, i.e. $q \in \alpha$.

3. If $p \in \alpha$, either $p \leq 0$ or $p > 0$. If $p \leq 0$ then (3) is satisfied with $r = 1$. If $p > 0$ and $p^2 < 2$ then (as shown in class) $r = \frac{2(p+1)}{p+2}$ satisfies $0 < p < r$ and $r^2 < 2$. Thus $r \in \alpha$, and (3) also holds in this case.

Step 2. Let $\mathbb{R}$ be the collection of all cuts of $\mathbb{Q}$. For $\alpha, \beta \in \mathbb{R}$ define $\alpha < \beta$ to mean α is a proper subset of β, (i.e. $\alpha \subset \beta$ but $\alpha \neq \beta$). $\mathbb{R}$ is an ordered set with relation $<$ defined above.

Step 3. *The ordered set $\mathbb{R}$ has the least-upper-bound property.* Let A be a nonempty subset of $\mathbb{R}$ which is bounded above. Let $\gamma = \bigcup_{\alpha \in A} \alpha$. Then $\gamma \in \mathbb{R}$ (i.e. γ satisfies (1), (2) and (3)), and $\gamma = \sup A$.

Step 4. If $\alpha, \beta \in \mathbb{R}$ define

$$\alpha + \beta = \{r + s : r \in \alpha \text{ and } s \in \beta\},$$

$$0^* = \{p \in \mathbb{Q} : p < 0\},$$

and

$$\alpha^* = \{p \in \mathbb{Q} : \text{there exits } r > 0 \text{ such that } -p - r \notin \alpha\}.$$

$\alpha + \beta$, 0^* and α^* are cuts. The axioms for addition hold in $\mathbb{R}$, with 0^* playing the role of 0, and α^* playing the role of $-\alpha$.

Step 5. After proving that the axioms of addition hold in $\mathbb{R}$ for the operation defined in Step 4, one can show using the cancellation law that

$$\text{If } \alpha, \beta, \gamma \in \mathbb{R} \text{ and } \beta < \gamma, \text{ then } \alpha + \beta < \alpha + \gamma.$$

Step 6. Initially we define multiplication for positive real numbers. Let $\mathbb{R}^+ = \{\alpha \in \mathbb{R} : \alpha > 0^{ast}\}$. If $\alpha, \beta \in \mathbb{R}^+$ define

$$\alpha\beta = \{p \in \mathbb{Q} : p \leq rs \text{ for some } r \in \alpha,\ s \in \beta,\ r > 0,\ s > 0\},$$

$$1^* = \{p \in \mathbb{Q} : p < 1\},$$

and

$$\alpha_* = \{p \in \mathbb{Q} : p \leq 0\} \cup \{p \in \mathbb{Q} :\ p > 0 \text{ and there exits } r > 0 \text{ such that } \frac{1}{p} - r \notin \alpha\}.$$

$\alpha\beta$, 1^* and α_* are cuts. The axioms for multiplication hold in $\mathbb{R}^+$, with 1^* playing the role of 1, and α_* playing the role of $\frac{1}{\alpha}$, for $\alpha > 0^*$. Note that if $\alpha > 0^*$ and $\beta > 0^*$ then $\alpha\beta > 0^*$. One also checks that the distributive law holds in $\mathbb{R}^+$.

Step 7. We complete the definition of multiplication by setting $\alpha 0^* = 0^*\alpha = 0^*$, and by setting

$$\alpha\beta = \begin{cases} (-\alpha)(-\beta) & \text{if} \quad \alpha < 0^*,\ \beta < 0^*, \\ -[(-\alpha)\beta] & \text{if} \quad \alpha < 0^*,\ \beta > 0^*, \\ -[\alpha(-\beta)] & \text{if} \quad \alpha > 0^*,\ \beta < 0^*. \end{cases}$$

The products on the right were defined in Step 6. Having checked the axioms of multiplication in $\mathbb{R}^+$ it is simple to prove them in $\mathbb{R}$ by repeated applications of the identity $\gamma = -(-\gamma)$. The proof of the distributive law is done by cases.

THIS COMPLETES THE SKETCH OF THE PROOF THAT $\mathbb{R}$ IS AN ORDERED FIELD WITH THE LEAST-UPPER-BOUND PROPERTY.

Step 8. We associate with each $r \in \mathbb{Q}$ the set

$$r^* = \{p \in \mathbb{Q} : p < r\}.$$

r^* is a (rational) cut, thus $r^* \in \mathbb{R}$. The rational cuts satisfy the following relations:

- $r^* + s^* = (r+s)^*$.
- $r^* s^* = (rs)^*$.
- $r^* < s^*$ if and only if $r < s$.

Step 9. Step 8 says that the rational numbers can be identified with the rational cuts. This identification preserves sums, products and order. Thus the ordered field $\mathbb{Q}$ is *isomorphic* to the ordered field $\mathbb{Q}^* \subset \mathbb{R}$ whose elements are the rational cuts.

THIS IDENTIFICATION OF $\mathbb{Q}$ WITH $\mathbb{Q}^*$ ALLOWS US TO REGARD $\mathbb{Q}$ AS A SUBFIELD OF $\mathbb{R}$.