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Besicovitch-type properties of measures and submanifolds

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1. Introduction

The motivation for this work comes from two different fields: Geometric Measure Theory and Differential Geometry.

To understand the first motivation, let us recall the following conjecture that greatly stimulated the development of the Geometric Measure Theory (cf. [3], 3. 3. 22). Let $n \leqq k$ be positive integers; denote by $B(x, r)$ the k-dimensional open ball in $\mathbb{R}^k$ with center x and radius $r > 0$, and by α_n the n-dimensional volume of a unit ball in $\mathbb{R}^n$.

Conjecture. *If Ψ is a Borel measure on $\mathbb{R}^k$ and*

$$\lim_{r \searrow 0} \frac{\Psi(B(x, r))}{\alpha_n r^n} = 1$$

holds Ψ-almost everywhere, then Ψ is an n-dimensional (Hausdorff) measure concentrated on a countable union $M \subset \mathbb{R}^k$ of smooth n-dimensional submanifolds: $\Psi = \mathcal{H}^n \llcorner M$.

Let us recall that it was already A.S. Besicovitch in [1] who settled a special case of the conjecture for $n = 1$ and $k = 2$. Thus we can refer well to the limit formula above as to the "Besicovitch Limit Property" (BLP). Other important special cases were proved in [5], [6], [7] and [8]. (See also [3] and [2].) Very recently, the second author has given the general proof of the above conjecture (to appear in Ann. Math.). The new method is based on an approximation of the given measure Ψ by its tangent measures. Here, each tangent measure Φ on $\mathbb{R}^k$ is known to satisfy a property which is more special than BLP, namely

$$\text{(GBP)} \quad \Phi(B(x, r)) = \alpha_n r^n \text{ for each } x \in \operatorname{spt} \Phi \text{ and each } r > 0.$$

(Here "GBP" stands for "Global Besicovitch Property".) Typical examples of measures satisfying the GBP are Hausdorff measures of n-dimensional linear subspaces in $\mathbb{R}^k$, but nonlinear examples are also known. A natural problem to solve (although this is not necessary for the proof of the original conjecture) is to give a complete list of measures satisfying the GBP.

The second motivation is connected with the work [9] of A. Gray and L. Vanhecke on the volume of geodesic balls in Riemannian manifolds. The basic "Volume Conjecture" says that a smooth Riemannian manifold in which every sufficiently small geodesic ball has the same volume as the Euclidean ball of the same dimension and radius must be locally flat. The "Volume Conjecture" has been proved only in some special cases (see also [11]).

Now, we can consider also an "extrinsic" analogue of this conjecture, namely the problem of classifying all n-dimensional smooth submanifolds M of the Euclidean space $\mathbb{R}^k$ with the following property: for each $x \in M$ and each sufficiently small $r>0$, the intersection of the k-dimensional ball $B(x,r)$ with M has the same n-dimensional (Riemannian) volume as the intersection of $B(x,r)$ with the tangent space $T_x M$. In other words, we want to classify the smooth n-dimensional submanifolds $M \subset \mathbb{R}^k$ with the property

(LBP) $\mathrm{Vol}(B(x,r)\cap M) = \alpha_n r^n$ *for each* $x \in M$ *and each sufficiently small* $r>0$.

(Here "LBP" stands for "Local Besicovitch Property".)

If we admit here an arbitrary $r>0$, we get just the GBP for submanifolds (or, what is the same, for the Hausdorff measures concentrated on submanifolds). Let us remark that the volumes $\mathrm{Vol}(B(x,r)\cap M)$ for arbitrary $r>0$ have been also studied, in a different context, by R. Osserman [12].

In this paper we solve explicitly the classification problem for GBP in the special case $k=n+1$ (codimension 1). We prove, a.o., that the classification problem for both measures and hypersurfaces is reduced to the classification of certain quadrics in $\mathbb{R}^{n+1}$. Then we can apply differential-geometric methods similar to those in [9] to make the last step of the classification. (The LBP problem for hypersurfaces is solved only in some very special cases, and little is known about the analogous LBP problem for measures.)

We state our main result in two versions:

Main Theorem (Measure-theoretic version). *Let Φ be a non-zero Borel measure fulfilling the* GBP *on $\mathbb{R}^{n+1}$. Then, after a rotation and a translation, either*

$$\Phi = \mathscr{H}^n \llcorner \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = 0\},$$

or

$$n \geqq 3, \text{ and } \Phi = \mathscr{H}^n \llcorner \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_4^2 = x_1^2 + x_2^2 + x_3^2\}.$$

Main Theorem (Geometric version). *Let $M \subset \mathbb{R}^{n+1}$ be a topologically embedded n-dimensional C^1-submanifold satisfying the* GBP. *Then, with respect to some Cartesian coordinate system, M is either an open subset of the hyperplane $\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = 0\}$, or one of the direct product $\mathbf{C}_1^3 \times \mathbb{R}^{n-3}$, where $\mathbf{C}_1^3$ is the cone $\{(x_1, \ldots, x_4) \in \mathbb{R}^4 \mid x_4^2 = x_1^2 + x_2^2 + x_3^2\}$. Moreover, the complement of M is a set of zero (n-dimensional Hausdorff) measure.*

The present paper is essentially an interdisciplinary one. To make the article easily accessible to differential geometers, we include at the beginning of section 4 a short information about the main measure-theoretic notions used here.

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2. Intersections of hypersurfaces with balls— a Taylor formula for the volume

We shall introduce for this section the following list of symbols (see [10] for more details).

Notation 2. 1. $\mathbb{R}^{n+1}$ ... the $(n+1)$-dimensional Euclidean space with the standard metric.

$M \subset \mathbb{R}^{n+1}$ ... a topologically embedded n-dimensional hypersurface.

$\vec{h} = (h_{ij})$ ... the second fundamental form of M.

$h = \operatorname{Trace} \vec{h}$ ... the (non-averaged) mean curvature of M.

g ... the induced Riemann metric on M.

∇ ... the Levi-Civita connection of M.

R ... the Riemann curvature tensor.

ϱ ... the Ricci form.

τ ... the scalar curvature.

Δ ... the Riemannian Laplacian.

$|| \ ||$ ... the length (norm) of a tensor with respect to the Riemannian inner product.

$B^{n+1}(m, r)$ (or $B(m, r)$) ... the $(n+1)$-dimensional open ball with center $m \in M$ and radius $r > 0$ in $\mathbb{R}^{n+1}$.

α_n ... the volume of the unit ball in $\mathbb{R}^n$.

$\mathcal{H}^n$ ... the n-dimensional Hausdorff measure (which on M coincides with the n-dimensional Riemann measure).

We are going to prove

Theorem 2. 2. *Let $M \subset \mathbb{R}^{n+1}$ be an n-dimensional hypersurface of class C^5. Then, for any $m \in M$ and for $r \searrow 0$, we have the following Taylor formula:*

$$(2.\ 1)\quad \mathcal{H}^n(M \cap B^{n+1}(m, r)) = \alpha_n r^n \left(1 + \frac{A(m)\, r^2}{8(n+2)} + \frac{B(m)\, r^4}{8(n+4)\,(n+2)} + O(r^5)\right)$$

where A, B are functions on M given by

$$(2.2)\quad \begin{cases} A = h^2 - 2\tau = 2\|\vec{h}\|^2 - h^2, \\ B = \dfrac{1}{16}h^4 - \dfrac{3}{4}h^2\|\vec{h}\|^2 + \dfrac{5}{4}\|\vec{h}\|^4 - \dfrac{1}{4}\|R\|^2 + \dfrac{1}{2}\|\varrho\|^2 + \Delta\|\vec{h}\|^2 \\ \qquad -\dfrac{2}{3}\|\nabla\vec{h}\|^2 - \dfrac{1}{2}h\Delta h. \end{cases}$$

The proof of Theorem 2. 2 will be given in several steps. We shall start with the calculation of *standard geometric invariants* (up to order 4) at a fixed point of M in terms of the parameters which depend on a special coordinate system. Then we derive the basic *integral formula* in terms of the same initial parameters, and finally, we *translate* our result back into the language of geometric invariants.

2a) Standard geometric invariants

Let us choose $m \in M$ and consider a (new) Cartesian coordinate system $(x_1, \dots, x_n, z)$ in $\mathbb{R}^{n+1}$ such that the origin is fixed at the point m, the z-axis coincides with the normal to M at m, and the x_i-axes lie in the principal directions of M at m. Then $(x_1, \dots, x_n)$ are local coordinates on M in a neighborhood of m. We can write, near the origin

$$(2.3)\quad z = \frac{1}{2}\sum \lambda_i x_i^2 + \frac{1}{6}\sum C_{ijk}x_i x_j x_k + \frac{1}{24}\sum D_{ijkl}x_i x_j x_k x_l + O(\varrho^5),$$

with symmetric coefficients, where

$$(2.4)\quad \varrho = \sqrt{\sum_{i=1}^{n} x_i^2},$$

and the summation in (2. 3) is extended to all *repeating* indices, ranging from 1 to n. (We shall use the same summation convention in many places throughout this section.)

We see that $\lambda_1, \dots, \lambda_n$ are the principal curvatures at m. The hypersurface M can be expressed locally as a graph

$$\vec{M}(x_1, \dots, x_n) = (x_1, \dots, x_n, z(x_1, \dots, x_n)).$$

For its partial derivatives we get

$$\vec{M}_i' = (0, \dots, 1, \dots, 0, z_i'), \quad i = 1, \dots, n$$

and the components of the Riemann metric g on M near the origin are given by

$$(2.5)\quad g_{ij} = \vec{M}_i' \cdot \vec{M}_j' = \delta_{ij} + z_i' z_j'.$$

Further, we consider a unit normal vector near the origin

$$\vec{n}=\frac{1}{\sqrt{\mathscr{D}}}(z_1',\ldots,z_n',-1),$$

where

$$\mathscr{D}=1+\sum_{i=1}^{n}(z_i')^2. \tag{2.6}$$

Then the components of the second fundamental form $\vec{h}$ are given by

$$h_{ij}=-\vec{M}_{ij}''\cdot\vec{n}=\frac{z_{ij}''}{\sqrt{\mathscr{D}}}. \tag{2.7}$$

From (2. 3) we have

$$z_i'=\lambda_i x_i+O(\varrho^2),\quad i=1,\ldots,n \tag{2.8}$$

and substituting into (2. 5), we obtain

$$g_{ij}=\delta_{ij}+\lambda_i\lambda_j x_i x_j+O(\varrho^3). \tag{2.9}$$

Hence

$$\begin{cases}(g_{ij})(m)=\delta_{ij},\ \left(\dfrac{\partial g_{ij}}{\partial x_k}\right)(m)=0,\\ \qquad\qquad\qquad\qquad\qquad\qquad (i,j,k,l=1,\ldots,n).\\ \left(\dfrac{\partial^2 g_{ij}}{\partial x_k\,\partial x_l}\right)(m)=(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\,\lambda_i\lambda_j,\end{cases} \tag{2.10}$$

We shall now calculate the components Γ_{ij}^k of the Levi-Civita connection and their first partial derivatives at m. Using the classical formula

$$\sum g_{lk}\Gamma_{ij}^l=\frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x_j}+\frac{\partial g_{jk}}{\partial x_i}-\frac{\partial g_{ij}}{\partial x_k}\right) \tag{2.11}$$

we get from (2. 10)

$$(\Gamma_{ij}^k)(m)=0\quad(i,j,k=1,\ldots,n). \tag{2.12}$$

Differentiating (2.11) and using (2. 10), (2. 12) we obtain

$$\begin{aligned}&\left(\frac{\partial\Gamma_{ii}^k}{\partial x_l}\right)(m)=\delta_{kl}\lambda_i\lambda_k,\ \left(\frac{\partial\Gamma_{ij}^k}{\partial x_l}\right)(m)=0\quad\text{for}\quad i\neq j,\\ &(i,j,k,l=1,\ldots,n).\end{aligned} \tag{2.13}$$

Next, we shall calculate the components of the curvature. Let us define, as in [10],

$$R(\partial_k, \partial_l)\,\partial_j = \sum R^i_{jkl}\partial_i,\quad R_{ijkl} = \sum g_{ia}R^a_{jkl}. \tag{2.14}$$

Then

$$R^i_{jkl} = \frac{\partial \Gamma^i_{lj}}{\partial x_k} - \frac{\partial \Gamma^i_{kj}}{\partial x_l} + \sum (\Gamma^m_{lj}\Gamma^i_{km} - \Gamma^m_{kj}\Gamma^i_{lm}), \tag{2.15}$$

and from (2. 12), (2. 13) we get

$$\begin{cases} (R_{ijij})(m) = \lambda_i\lambda_j \quad (i \neq j), \\ (R_{ijkl})(m) = 0 \quad \text{if at least three indices among } i, j, k, l \\ \qquad\qquad\quad \text{are distinct.} \end{cases} \tag{2.16}$$

Because the coordinate frame at m is an orthonormal basis, we get

$$\|R\|^2(m) = 4 \sum_{1 \leq i < j \leq n} (\lambda_i\lambda_j)^2 = 2\left(\left(\sum_{i=1}^n \lambda_i^2\right)^2 - \sum_{i=1}^n \lambda_i^4\right). \tag{2.17}$$

Now, due to (2. 6)—(2. 8) we have (as expected)

$$(h_{ii})(m) = \lambda_i,\ (h_{ij})(m) = 0 \quad (i \neq j);\quad i, j = 1, \ldots, n. \tag{2.18}$$

For the mean curvature and the length of the second fundamental form we hence get

$$h(m) = \sum_{i=1}^n \lambda_i,\quad \|\vec{h}\|^2(m) = \sum_{i=1}^n \lambda_i^2. \tag{2.19}$$

Finally, using (2. 16)—(2. 19), we can calculate the length of the curvature tensor R, the Ricci form ϱ, and the scalar curvature τ as follows:

$$\|R\|^2(m) = 2\left(\|\vec{h}\|^4 - \sum_{i=1}^n \lambda_i^4\right)(m), \tag{2.20}$$

$$\begin{aligned} \|\varrho\|^2(m) &= \sum_{i=1}^n \varrho_{ii}^2 = \sum_{i=1}^n \left(\sum_{j=1}^n R_{ijij}\right)^2 = \sum_{i=1}^n (h\lambda_i - \lambda_i^2)^2 \\ &= \left(h^2\|\vec{h}\|^2 + \sum_{i=1}^n \lambda_i^4 - 2h\sum_{i=1}^n \lambda_i^3\right)(m), \end{aligned} \tag{2.21}$$

$$\tau(m) = \sum \varrho_{ii} = \sum_{i=1}^n (h\lambda_i - \lambda_i^2) = (h^2 - \|\vec{h}\|^2)(m). \tag{2.22}$$

Further, differentiating (2. 3) in higher degrees, we have

$$z''_{ij} = \lambda_i\delta_{ij} + \sum C_{ijk}x_k + \frac{1}{2}\sum D_{ijkl}x_kx_l + O(\varrho^3).$$

Using (2. 6) and (2. 8) we obtain

$$\frac{1}{\sqrt{\mathscr{D}}} = 1 - \frac{1}{2}\sum_{i=1}^{n}(z_i')^2 + \cdots = 1 - \frac{1}{2}\sum \lambda_k^2 x_k^2 + O(\varrho^3),$$

and using also (2. 7) we get

$$h_{ij} = \lambda_i \delta_{ij} + \sum C_{ijk} x_k + \frac{1}{2}\sum D_{ijkl} x_k x_l - \frac{1}{2}\lambda_i \delta_{ij} \sum \lambda_k^2 x_k^2 + O(\varrho^3). \tag{2. 23}$$

According to the standard formula

$$\nabla_k h_{ij} = \partial_k h_{ij} - \sum (\Gamma_{ki}^u h_{uj} + \Gamma_{kj}^u h_{iu}) \tag{2. 24}$$

and (2. 12), we derive at m

$$\begin{aligned}(\nabla_{kk}^2 h_{ij})(m) &= (\partial_k(\nabla_k h_{ij}))(m)\\ &= (\partial_{kk}^2 h_{ij})(m) - \sum_{u=1}^{n}\left(\frac{\partial \Gamma_{ki}^u}{\partial x_k} h_{uj} + \frac{\partial \Gamma_{kj}^u}{\partial x_k} h_{iu}\right)(m)\\ &= D_{ijkk} - \lambda_i \lambda_k^2 \delta_{ij} - \left(\frac{\partial \Gamma_{ki}^j}{\partial x_k}\right)(m)\,\lambda_j - \left(\frac{\partial \Gamma_{kj}^i}{\partial x_k}\right)(m)\,\lambda_i.\end{aligned} \tag{2. 25}$$

Using also (2. 13), we obtain from (2. 23)—(2. 25)

$$(\nabla_k h_{ij})(m) = (\partial_k h_{ij})(m) = C_{ijk}, \tag{2. 26}$$

$$\begin{aligned}(\nabla_{kk}^2 h_{ii})(m) &= D_{iikk} - \lambda_i \lambda_k^2 \quad (i \neq k),\\ (\nabla_{ii}^2 h_{ii})(m) &= D_{iiii} - 3\lambda_i^3.\end{aligned} \tag{2. 27}$$

Finally, using the formula for the Laplacian

$$\Delta = \sum g^{ij} \nabla_{ij}^2,$$

we get first

$$\begin{aligned}(\Delta h_{ii})(m) &= \left(\sum_{k=1}^{n} \nabla_{kk}^2 h_{ii}\right)(m)\\ &= \sum_{k=1}^{n} D_{iikk} - \lambda_i \|\vec{h}\|^2(m) - 2\lambda_i^3, \quad (i = 1, \ldots, n),\end{aligned}$$

$$\begin{aligned}\langle \Delta \vec{h}, \vec{h}\rangle(m) &= \sum (h_{ij} \Delta h_{ij})(m)\\ &= \sum_{i=1}^{n} \lambda_i (\Delta h_{ii})(m)\\ &= \sum_{i,k=1}^{n} \lambda_i D_{iikk} - \|\vec{h}\|^4(m) - 2\sum_{i=1}^{n} \lambda_i^4,\end{aligned}$$

and hence, using also (2. 20), we obtain

$$\langle \Delta \vec{h}, \vec{h} \rangle (m) = \sum_{i,j=1}^{n} \lambda_i D_{iijj} - 3\|\vec{h}\|^4(m) + \|R\|^2(m). \tag{2. 28}$$

Also, we have $h = \sum g^{ij} h_{ij}$ and hence

$$(\Delta h)(m) = \sum (\Delta h_{ii})(m) = \sum_{i,k=1}^{n} D_{iikk} - h(m)\|\vec{h}\|^2(m) - 2\sum_{i=1}^{n} \lambda_i^3. \tag{2. 29}$$

Using (2. 20), (2. 21) we get

$$(h\Delta h)(m) = \left\{ h \sum_{i,j=1}^{n} D_{iijj} - 2h^2\|\vec{h}\|^2 - \|\vec{h}\|^4 + \|\varrho\|^2 + \frac{1}{2}\|R\|^2 \right\}(m). \tag{2. 30}$$

2b) The basic integral formula

Let us now express the volume of $B(m, r) \cap M$. In the notation of (2. 6) we have, for any sufficiently small $r > 0$,

$$\mathscr{H}^n(B^{n+1}(m, r) \cap M) = \int_{\Omega(r)} \sqrt{\mathscr{D}}\, dx_1 \ldots dx_n, \tag{2. 31}$$

where $\Omega(r)$ is the orthogonal projection of $B(m, r) \cap M$ on the hyperplane

$$\mathbb{R}^n \cong \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}.$$

Let us introduce spherical coordinates in $\mathbb{R}^n$, $(\varrho, a_1, \ldots, a_n)$, by

$$x_i = \varrho a_i,\ i = 1, \ldots, n,\ \sum_{i=1}^{n} a_i^2 = 1. \tag{2. 32}$$

This means that each point $a = (a_1, \ldots, a_n)$ belongs to the unit sphere $S^{n-1}(1) = S^{n-1}$ in $\mathbb{R}^n$, and ϱ is given by (2. 4).

We shall need the following

Lemma 2. 3. *For any sufficiently small $r > 0$, the domain $\Omega(r)$ is given by*

$$\Omega(r) = \{(\varrho, a) | 0 \leqq \varrho < P(a, r)\}, \tag{2. 33}$$

where $P(a, r)$ is the unique solution of the equation

$$\varrho^2 + z(\varrho a)^2 = r^2,\ \varrho \in [0, r].$$

Proof. It suffices to note that $\Omega(r)$ is a sublevel set of the function $\|\vec{M}(x)\|^2 = \sum x_i^2 + z(x)^2$ which is strictly convex in a neighborhood of the origin, because

$$\partial_i \partial_j (\varrho^2 + z^2) = 2\delta_{ij} + O(\varrho^2)$$

by (2. 3).

Using our spherical coordinates and Lemma 2. 3, we can transform the integral in (2. 31) to the form

$$\text{(2. 34)} \quad \mathscr{H}^n(B^{n+1}(m, r) \cap M) = \int_{S^{n-1}} d\sigma \left(\int_0^{P(a,r)} \sqrt{\mathscr{D}(\varrho, a_1, \ldots, a_n)}\, \varrho^{n-1}\, d\varrho \right)$$

where $d\sigma$ denotes the volume element on $S^{n-1}(1) = S^{n-1}$. In particular, we shall denote

$$\text{(2. 35)} \quad \omega_{n-1} = \int_{S^{n-1}} d\sigma = \text{the } (n-1)\text{-dimensional volume of } S^{n-1}(1).$$

Near the origin, we can write

$$\text{(2. 36)} \qquad z = B\varrho^2 + C\varrho^3 + D\varrho^4 + O(\varrho^5),$$

where, according to (2. 3),

$$\text{(2. 37)} \quad B = \frac{1}{2} \sum \lambda_i a_i^2, \quad C = \frac{1}{6} \sum C_{ijk} a_i a_j a_k, \quad D = \frac{1}{24} \sum D_{ijkl} a_i a_j a_k a_l.$$

Further, we have from (2. 3)

$$\text{(2. 38)} \qquad z_i' = \lambda_i x_i + \frac{1}{2} \sum C_{ijk} x_j x_k + \frac{1}{6} \sum D_{ijkl} x_j x_k x_l + O(\varrho^4)$$

and hence

$$\text{(2. 39)} \qquad \frac{1}{2} \sum_{i=1}^{n} (z_i')^2 = \bar{B}\varrho^2 + \bar{C}\varrho^3 + \bar{D}\varrho^4 + O(\varrho^5),$$

where

$$\text{(2. 40)} \quad \begin{aligned} \bar{B} &= \frac{1}{2} \sum_{i=1}^{n} \lambda_i^2 a_i^2, \quad \bar{C} = \frac{1}{2} \sum_{i,j,k=1}^{n} \lambda_i C_{ijk} a_i a_j a_k, \\ \bar{D} &= \frac{1}{6} \sum_{i,j,k,l=1}^{n} \lambda_i D_{ijkl} a_i a_j a_k a_l + \frac{1}{8} \sum_{i=1}^{n} \left(\sum_{j,k=1}^{n} C_{ijk} a_j a_k \right)^2. \end{aligned}$$

Using (2. 39) we get

$$\begin{aligned}\sqrt{\mathcal{D}(\varrho, a)} &= \sqrt{1+\sum_{i=1}^{n}(z_i')^2} \\ &= 1+\frac{1}{2}\sum_{i=1}^{n}(z_i')^2-\frac{1}{8}\left(\sum_{i=1}^{n}(z_i')^2\right)^2+O(\varrho^6) \\ &= 1+\bar{B}\varrho^2+\bar{C}\varrho^3+\left(\bar{D}-\frac{1}{2}\bar{B}^2\right)\varrho^4+O(\varrho^5).\end{aligned}$$

Hence (2. 34) takes on the form

$$\begin{aligned}&\mathcal{H}^n(B^{n+1}(m, r)\cap M) \\ &= \int_{S^{n-1}} d\sigma\left\{\int_0^{P(a,r)}\left[\varrho^{n-1}+\bar{B}\varrho^{n+1}+\bar{C}\varrho^{n+2}+\left(\bar{D}-\frac{1}{2}\bar{B}^2\right)\varrho^{n+3}+O(\varrho^{n+4})\right]d\varrho\right\} \\ &= \int_{S^{n-1}} d\sigma\left[\frac{\varrho^n}{n}+\bar{B}\frac{\varrho^{n+2}}{n+2}+\bar{C}\frac{\varrho^{n+3}}{n+3}+\left(\bar{D}-\frac{1}{2}\bar{B}^2\right)\frac{\varrho^{n+4}}{n+4}+O(\varrho^{n+5})\right]_{\varrho=0}^{\varrho=P(a,r)}.\end{aligned} \tag{2. 41}$$

We shall now estimate the bound $P(a, r)$. According to Lemma 2. 3, this is the unique solution of the equation

$$\varrho^2+(z(\varrho a))^2=r^2 \quad \text{in} \quad [0, r].$$

By (2. 36), this equation takes on the form

$$\varrho^2+B^2\varrho^4+2BC\varrho^5+(C^2+2BD)\varrho^6+O(\varrho^7)=r^2. \tag{2. 42}$$

For any fixed $a=(a_1, \ldots, a_n)\in S^{n-1}$, and any small $r>0$, the solution $\varrho=P(a, r)$ of (2. 42) is given by

$$\varrho=r-\frac{B^2}{2}r^3-BCr^4+\left(\frac{7}{8}B^4-\frac{1}{2}C^2-BD\right)r^5+O(r^6), \tag{2. 43}$$

as follows by a direct check.

Substituting (2. 43) into (2. 41), we get by a routine calculation

$$\begin{aligned}&\mathcal{H}^n(B^{n+1}(m, r)\cap M) \\ &= \int_{S^{n-1}}\left[\frac{r^n}{n}+\left(\frac{\bar{B}}{n+2}-\frac{B^2}{2}\right)r^{n+2}+\left(\frac{\bar{C}}{n+3}-BC\right)r^{n+3}\right. \\ &\quad \left.+\left(\frac{n+6}{8}B^4-\frac{1}{2}B^2\bar{B}-\frac{1}{2}C^2-BD+\frac{1}{n+4}\left(\bar{D}-\frac{1}{2}\bar{B}^2\right)\right)r^{n+4}+O(r^{n+5})\right]d\sigma,\end{aligned} \tag{2. 44}$$

which is the wanted integral formula.

2c) Geometric evaluation of the integral formula

We have first

$$\int_{S^{n-1}} \frac{r^n}{n}\, d\sigma = \frac{\omega_{n-1}}{n} r^n = \alpha_n r^n. \tag{2.45}$$

To integrate the other functions over the unit sphere in (2.44), we can use, e.g., the formula in [3], 3.2.13, or the Pizzetti formula (see [13]).

By a routine calculation we get the following nonzero integrals for the monomials of various types of degrees 2, 4, 6 and 8:

$$\int a_i^2 = \alpha_n, \quad \int a_i^4 = \frac{3\alpha_n}{n+2}, \quad \int a_i^2 a_j^2 = \frac{\alpha_n}{n+2} \quad \text{for} \quad i \neq j,$$

$$\int a_i^6 = \frac{15\alpha_n}{(n+4)(n+2)}, \quad \int a_i^4 a_j^2 = \frac{3\alpha_n}{(n+4)(n+2)} \quad \text{for} \quad i \neq j,$$

$$\int a_i^2 a_j^2 a_k^2 = \frac{\alpha_n}{(n+4)(n+2)} \quad \text{for} \quad i \neq j \neq k \neq i,$$

$$\int a_i^8 = \frac{105\alpha_n}{(n+6)(n+4)(n+2)}, \quad \int a_i^6 a_j^2 = \frac{15\alpha_n}{(n+6)(n+4)(n+2)} \quad \text{for} \quad i \neq j,$$

$$\int a_i^4 a_j^4 = \frac{9\alpha_n}{(n+6)(n+4)(n+2)} \quad \text{for} \quad i \neq j,$$

$$\int a_i^4 a_j^2 a_k^2 = \frac{3\alpha_n}{(n+6)(n+4)(n+2)} \quad \text{for} \quad i \neq j \neq k \neq i,$$

$$\int a_i^2 a_j^2 a_k^2 a_l^2 = \frac{\alpha_n}{(n+6)(n+4)(n+2)} \quad \text{for} \quad i, j, k, l \text{ all distinct.}$$

Now, we shall calculate (2.44) part by part. Using (2.37) and (2.40), we get

$$\int_{S^{n-1}} \left(\frac{\bar{B}}{n+2} - \frac{B^2}{2}\right) d\sigma = \frac{1}{2(n+2)} \left(\sum_{i=1}^n \lambda_i^2 \int a_i^2\right) - \frac{1}{8} \sum_{i=1}^n \lambda_i^2 \int a_i^4 - \frac{1}{8} \sum_{\substack{i,j=1 \\ (i \neq j)}}^n \lambda_i \lambda_j \int a_i^2 a_j^2$$

$$= \frac{\alpha_n}{8(n+2)} \left(\sum_{i=1}^n \lambda_i^2 - \sum_{\substack{i,j=1 \\ (i \neq j)}}^n \lambda_i \lambda_j\right).$$

Here the bracket is equal to

$$2 \sum_{i=1}^n \lambda_i^2 - \left(\sum_{i=1}^n \lambda_i\right)^2 = (2\|\vec{h}\|^2 - h^2)(m) = (h^2 - 2\tau)(m)$$

due to (2.19), (2.22). Finally, we get

$$r^{n+2} \int_{S^{n-1}} \left(\frac{\bar{B}}{n+2} - \frac{B^2}{2}\right) d\sigma = \alpha_n r^n \frac{A(m) r^2}{8(n+2)}$$

where $A(m) = (2\|\vec{h}\|^2 - h^2)(m) = (h^2 - 2\tau)(m)$.

This proves the first part of (2. 2).

Next, we have

$$\int_{S^{n-1}} \left(\frac{\bar{C}}{n+3} - BC \right) d\sigma = 0,$$

because we integrate a polynomial of odd order. Thus the coefficient of r^3 in formula (2. 1) is zero.

It remains to integrate the next term in (2. 44). Using our integral formulas for the monomials, and (2. 17), (2. 19), we get first by a lengthy but routine calculation

$$\begin{aligned}
&\int_{S^{n-1}} \left(\frac{n+6}{8} B^4 - \frac{1}{2} B^2 \bar{B} - \frac{1}{2(n+4)} \bar{B}^2 \right) d\sigma \\
&= \frac{\alpha_n}{8(n+4)(n+2)} \cdot \frac{1}{16} \Bigg(-63 \sum_{i=1}^{n} \lambda_i^4 - 13 \sum_{\substack{i,j=1 \\ (i \neq j)}}^{n} \lambda_i^2 \lambda_j^2 + 12 \sum_{\substack{i,j=1 \\ (i \neq j)}}^{n} \lambda_i^3 \lambda_j \\
&\quad + 10 \sum_{\substack{i,j,k=1 \\ (i \neq j \neq k \neq i)}}^{n} \lambda_i^2 \lambda_j \lambda_k + \sum_{\substack{i,j,k,l=1 \\ (i,j,k,l \\ \text{distinct})}}^{n} \lambda_i \lambda_j \lambda_k \lambda_l \Bigg) \\
&= \frac{\alpha_n}{8(n+4)(n+2)} \left(\frac{1}{16} h^4 + \frac{1}{4} h^2 \|\vec{h}\|^2 - \frac{17}{4} \|\vec{h}\|^4 + \frac{3}{2} \|R\|^2 \right) (m).
\end{aligned}$$

Further, we get

$$\begin{aligned}
(2.46) \quad \int_{S^{n-1}} C^2 d\sigma = \frac{\alpha_n}{36(n+4)(n+2)} \Bigg\{ &15 \sum_{i=1}^{n} (C_{iii})^2 + 27 \sum_{\substack{i,j=1 \\ (i \neq j)}}^{n} (C_{iij})^2 \\
&+ 18 \sum_{\substack{i,j=1 \\ (i \neq j)}}^{n} C_{iii} C_{ijj} + \sum_{\substack{i,j,k=1 \\ (i \neq j \neq k \neq i)}}^{n} (9 C_{iij} C_{kkj} + 6 (C_{ijk})^2) \Bigg\}.
\end{aligned}$$

In (2. 40), we put $\bar{D} = \bar{D}_1 + \bar{D}_2$, where

$$(2.47) \qquad \begin{cases} \bar{D}_1 = \dfrac{1}{6} \displaystyle\sum_{i,j,k,l=1}^{n} \lambda_i D_{ijkl} a_i a_j a_k a_l, \\ \bar{D}_2 = \dfrac{1}{8} \displaystyle\sum_{i=1}^{n} \left(\sum_{j,k=1}^{n} C_{ijk} a_j a_k \right)^2. \end{cases}$$

Then

$$\begin{aligned}
(2.48) \quad \int_{S^{n-1}} \bar{D}_2 d\sigma = \frac{\alpha_n}{8(n+2)} \Bigg\{ &3 \sum_{i=1}^{n} (C_{iii})^2 + 7 \sum_{\substack{i,j=1 \\ (i \neq j)}}^{n} (C_{ijj})^2 \\
&+ 2 \sum_{\substack{i,j=1 \\ (i \neq j)}}^{n} C_{iii} C_{ijj} + \sum_{\substack{i,j,k=1 \\ (i \neq j \neq k \neq i)}}^{n} (C_{ijj} C_{ikk} + 2 (C_{ijk})^2) \Bigg\}.
\end{aligned}$$

From (2. 46) and (2. 48) we derive

$$\int_{S^{n-1}} \left(\frac{1}{n+4}\bar{D}_2 - \frac{1}{2}C^2\right) d\sigma$$
$$= \frac{\alpha_n}{72(n+4)(n+2)}\left(12\sum_{i=1}^{n}(C_{iii})^2 + 36\sum_{\substack{i,j=1\\(i\neq j)}}^{n}(C_{ijj})^2 + 12\sum_{\substack{i,j,k=1\\(i\neq j\neq k\neq i)}}^{n}(C_{ijk})^2\right)$$
$$= \frac{\alpha_n}{6(n+4)(n+2)}\sum_{i,j,k=1}^{n}(C_{ijk})^2 = \frac{\alpha_n}{6(n+4)(n+2)}\|\nabla\vec{h}\|^2(m),$$

when using also (2. 26).

It remains to calculate the integral $\int\left(\frac{1}{n+4}\bar{D}_1 - BD\right)d\sigma$. We get first

$$\frac{1}{n+4}\int_{S^{n-1}}\bar{D}_1\,d\sigma = \frac{\alpha_n}{2(n+4)(n+2)}\left(\sum_{i=1}^{n}\lambda_i D_{iiii} + \sum_{\substack{i,j=1\\(i\neq j)}}^{n}\lambda_i D_{iijj}\right),$$

$$\int_{S^{n-1}} BD\,d\sigma = \frac{\alpha_n}{48(n+4)(n+2)}\left(15\sum_{i=1}^{n}\lambda_i D_{iiii} + 3\sum_{\substack{i,j=1\\(i\neq j)}}^{n}\lambda_j D_{iiii}\right.$$
$$\left. + 18\sum_{\substack{i,j=1\\(i\neq j)}}^{n}\lambda_i D_{iijj} + 3\sum_{\substack{i,j,k=1\\(i\neq j\neq k\neq i)}}^{n}\lambda_k D_{iijj}\right),$$

and hence

$$\int_{S^{n-1}}\left(\frac{1}{n+4}\bar{D}_1 - BD\right)d\sigma$$
$$= \frac{\alpha_n}{48(n+4)(n+2)}\left(9\sum_{i=1}^{n}\lambda_i D_{iiii} + \sum_{\substack{i,j=1\\(i\neq j)}}^{n}(6\lambda_i D_{iijj} - 3\lambda_j D_{iiii}) - 3\sum_{\substack{i,j,k=1\\(i\neq j\neq k\neq i)}}^{n}\lambda_k D_{iijj}\right)$$
$$= \frac{\alpha_n}{(n+4)(n+2)}\left(\frac{1}{4}\sum_{i,j=1}^{n}\lambda_i D_{iijj} - \frac{1}{16}\sum_{i,j,k=1}^{n}\lambda_k D_{iijj}\right)$$
$$= \frac{\alpha_n}{(n+4)(n+2)}\left(\frac{1}{4}\sum_{i,j=1}^{n}\lambda_i D_{iijj} - \frac{h(m)}{16}\sum_{i,j=1}^{n}D_{iijj}\right).$$

Using (2. 28), (2. 30), we obtain

$$\int_{S^{n-1}}\left(\frac{1}{n+4}\bar{D}_1 - BD\right)d\sigma$$
$$= \frac{\alpha_n}{8(n+4)(n+2)}\left(2\langle\Delta\vec{h},\vec{h}\rangle - \frac{1}{2}h\,\Delta h + \frac{11}{2}\|\vec{h}\|^4 - h^2\|\vec{h}\|^2 - \frac{7}{4}\|R\|^2 + \frac{1}{2}\|Q\|^2\right)(m).$$

Let us recall the obvious identity

$$2\langle \Delta \vec{h}, \vec{h} \rangle = \Delta \|\vec{h}\|^2 - 2\|\nabla \vec{h}\|^2. \tag{2.49}$$

Summing up our partial results and using (2. 49), we get at once

$$r^{n+4} \int_{S^{n-1}} \left(\frac{n+6}{8} B^4 - \frac{1}{2} B^2 \bar{B} - \frac{1}{2} C^2 - BD + \frac{1}{n+4} \left(\bar{D} - \frac{1}{2} \bar{B}^2 \right) \right) d\sigma$$
$$= \frac{\alpha_n r^{n+4}}{8(n+4)(n+2)} \left(\frac{1}{16} h^4 - \frac{3}{4} h^2 \|\vec{h}\|^2 + \frac{5}{4} \|\vec{h}\|^4 - \frac{1}{4} \|R\|^2 + \frac{1}{2} \|\varrho\|^2 \right.$$
$$\left. + \Delta \|\vec{h}\|^2 - \frac{2}{3} \|\nabla \vec{h}\|^2 - \frac{1}{2} h \Delta h \right)(m).$$

This completes the proof of formula (2. 2) and Theorem 2. 2.

3. The geometric classification problem for LBP and examples

We start with the following consequence of Theorem 2. 2.

Corollary 3. 1. *If a hypersurface $M \subset \mathbb{R}^{n+1}$ of class C^5 satisfies the* LBP, *than we have along M*

$$h^2 = 2\|\vec{h}\|^2 = 2\tau, \tag{3.1}$$

$$\frac{1}{2} \|\varrho\|^2 - \frac{1}{4} \|R\|^2 + \frac{1}{2} h \Delta h + \|dh\|^2 - \frac{2}{3} \|\nabla \vec{h}\|^2 = 0. \tag{3.2}$$

If a hypersurface of class C^3 satisfies the LBP, *then it still satisfies the condition* (3. 1).

Proof. (3. 1) is obvious for the class C^5, and we see from the proof of Theorem 2. 2 that it still holds for the class C^3. Now, (3. 2) can be obtained by the substitutions

$$\|\vec{h}\|^2 = \frac{1}{2} h^2, \quad \Delta \|\vec{h}\|^2 = \frac{1}{2} \Delta h^2 = h \Delta h + \|dh\|^2$$

into the condition $B = 0$ in (2. 2).

Now, the geometric classification problem for LBP can be solved easily when $\dim M \leqq 2$.

Theorem 3.2. a) *A connected 1-dimensional submanifold $M \subset \mathbb{R}^2$ of class C^1 satisfies the* LBP *iff it is an open line segment.*

b) *A connected 2-dimensional surface $M \subset \mathbb{R}^3$ of class C^3 satisfies the* LBP *iff it is an open subset of a plane, or of a sphere.*

Proof. For the dimension $n=1$ the result follows directly from the LBP and from the extremal property of the line segments. For $n=2$, using (3. 1) and (2. 19) we see that the principal curvatures λ_1, λ_2 are equal along M. Hence all points of M are umbilic and M must be a part of a plane, or of a sphere. An elementary calculation shows that, conversely, any 2-dimensional sphere satisfies the LBP.

For the higher dimensions, the local classification problem seems to be rather difficult. The following two theorems give a partial information:

Theorem 3. 3. *If $M_n \subset \mathbb{R}^{n+1}$ of class C^3 satisfies the* LBP, *then M_n has non-negative scalar curvature. If, in addition, the scalar curvature vanishes, or M_n is minimal, then each component of M_n is a part of a hyperplane.*

Proof. Obvious from formula (3. 1).

Theorem 3.4 *Let $M_3 \subset \mathbb{R}^4$ be a connected hypersurface of class C^5 with parallel second fundamental form. Then M_3 satisfies the* LBP *iff it is congruent to an open subset of a hyperplane, or of a cylinder $S^2(r) \times \mathbb{R}$.*

Proof. For the Weyl tensor C on M_3 we have $\|C\|^2 = \|R\|^2 - 4\|\varrho\|^2 + \tau^2 = 0$ (cf. [9], p. 173), and because $\nabla \vec{h} = 0$, the equation (3. 2) is reduced to $\|\varrho\|^2 = \frac{1}{2}\|R\|^2$. Hence we get

$$\|\varrho\|^2 = \frac{1}{2}\tau^2. \tag{3. 3}$$

Using (2. 16) we obtain

$$\begin{aligned} &\varrho_{11} = \lambda_1(\lambda_2+\lambda_3), \quad \varrho_{22} = \lambda_2(\lambda_1+\lambda_3), \quad \varrho_{33} = \lambda_3(\lambda_1+\lambda_2), \\ &\tau = 2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3). \end{aligned} \tag{3. 4}$$

Substituting (3. 4) into (3. 3) we get easily $(\lambda_1+\lambda_2+\lambda_3)\lambda_1\lambda_2\lambda_3 = 0$. If now $h = (\lambda_1+\lambda_2+\lambda_3) = 0$, we have a part of a hyperplane (Theorem 3. 3). Next, suppose $h \neq 0$, then $\lambda_1\lambda_2\lambda_3 = 0$ and hence (say) $\lambda_1 = 0$. Then (3. 1) implies $\lambda_2 = \lambda_3 = \frac{1}{2}h \neq 0$. Because $\vec{h}$ is parallel, we get $\lambda_2 = \lambda_3 = \text{const}$. Hence we see, by a standard argument, that M_3 is congruent to a part of the direct product $S^2(r) \times \mathbb{R}$, where $r^2 = \lambda_2^{-2} = \lambda_3^{-2}$.

The fact that $S^2(r) \times \mathbb{R}$ satisfies the LBP follows from Theorem 3. 2 and Theorem 3. 10 (see later in this section).

As a consequence, we see that a sphere $S^3(r)$ does not satisfy the LBP. But the 3-dimensional case is still non-trivial. In what follows, we shall give examples of 3-dimensional hypersurfaces in $\mathbb{R}^4$, satisfying the LBP, which are

(i) *non-homogeneous as Riemannian manifolds*, and

(ii) *embedded in $\mathbb{R}^4$ with non-constant mean curvature.*

Because our examples involve non-trivial computations, we shall start with some technical lemmas. In what follows, $\mathscr{H}^n$ will again denote the n-dimensional Hausdorff measure in $\mathbb{R}^d$ (whose restriction to a submanifold $M_n \subset \mathbb{R}^d$ coincides with the n-dimensional Riemann measure), and $\mathscr{L}^n$ will mean the n-dimensional Lebesgue measure in $\mathbb{R}^n$.

Lemma 3. 5. *Let $x \in \mathbb{R}^3$ be a point and $\|x\|$ its Euclidean norm. Then, for any $s>0$, $t>0$, the 2-dimensional area $\mathscr{H}^2(S^2(o,t) \cap B^3(x,s))$ equals to*

(i) $\pi\left(2t^2 - \frac{t}{\|x\|}(\|x\|^2 + t^2 - s^2)\right)$ *if* $\|x\| \neq 0$, $|t-s| \leqq \|x\| \leqq t+s$,

(ii) $4\pi t^2$ *if* $\|x\| \leqq s-t$,

(iii) 0 *if* $\|x\| \geqq t+s$, or $\|x\| \leqq t-s$.

Proof. We use the spherical coordinates on $S^2(o,t)$, and then the calculation is straightforward.

Lemma 3. 6. *Consider the 3-dimensional cone in $\mathbb{R}^4$,*

$$\mathbf{C} = \{u \in \mathbb{R}^4 \mid u_4^2 = a(u_1^2 + u_2^2 + u_3^2)\} \tag{3. 5}$$

where $a>0$ is a fixed parameter. Let $\mathbb{R}^3 \subset \mathbb{R}^4$ be the hyperplane given by $u_4 = 0$, and $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ the orthogonal projection. For a point $m \in \mathbf{C}$, put $x = \pi(m) \in \mathbb{R}^3$. Then, for any $r>0$,

$$\mathscr{H}^3(B^4(m,r) \cap \mathbf{C}) = \sqrt{1+a}\,(\mathscr{L}^3(E_1) + \mathscr{L}^3(E_2)), \tag{3. 6}$$

where

$$\begin{aligned} E_1 &= \{z \in \mathbb{R}^3 \mid \|z-x\|^2 + a(\|z\| - \|x\|)^2 \leqq r^2\}, \\ E_2 &= \{z \in \mathbb{R}^3 \mid \|z-x\|^2 + a(\|z\| + \|x\|)^2 \leqq r^2\}, \end{aligned} \tag{3. 7}$$

and $\|\ \|$ denotes the norm in $\mathbb{R}^3$.

Proof. Consider the half-cones in $\mathbb{R}^4$ composing $\mathbf{C}$:

$$\begin{aligned} \mathbf{C}_1 &= \{u \in \mathbb{R}^4 \mid u_4 = \sqrt{a}\|z\|\}, \\ \mathbf{C}_2 &= \{u \in \mathbb{R}^4 \mid u_4 = -\sqrt{a}\|z\|\}, \quad z = (u_1, u_2, u_3) \in \mathbb{R}^3. \end{aligned} \tag{3. 8}$$

We can assume that $m \in \mathbf{C}_1$; the other case is similar. We see easily that

$$E_i = \pi(B^4(m,r) \cap \mathbf{C}_i), \quad i = 1,2. \tag{3. 9}$$

By the standard formula of the calculus hence we get for $i = 1,2$,

$$\mathscr{H}^3(B^4(m,r) \cap \mathbf{C}_i) = \int_{E_i} \sqrt{1+a}\, du_1\, du_2\, du_3 = \sqrt{1+a}\, \mathscr{L}^3(E_i). \tag{3. 10}$$

This completes the proof.

Let us notice that, for $m \in \mathbf{C}_2$, the role of E_1 and E_2 in the formulas (3. 9), (3. 10) interchanges.

Proposition 3. 7. *The manifold* $\mathbf{C}\setminus\{o\}$ *satisfies the* LBP *for every value of the parameter* $a>0$.

Proof. We show the LBP for the component $\mathbf{C}_1\setminus\{o\}$. Let $m \in \mathbf{C}_1\setminus\{o\}$, $x=\pi(m)$, and choose

$$r<\min\{\|x\|, \sqrt{a}\|x\|\}. \tag{3. 11}$$

We first see that $E_2=\emptyset$. In fact, for $z \in E_2$ we get $a\|x\|^2 \leqq r^2$, which contradicts to (3. 11). Further, for any $t \in \left[\|x\|-\frac{r}{\sqrt{a}}, \|x\|+\frac{r}{\sqrt{a}}\right]$

$$S^2(o,t)\cap E_1 = S^2(o,t)\cap B^3(x,s), \text{ where } s=\sqrt{r^2-a(t-\|x\|)^2}. \tag{3. 12}$$

On the other hand, $S^2(o,t)\cap E_1=\emptyset$ for $|t-\|x\||<\frac{r}{\sqrt{a}}$. Moreover,

$$\mathscr{H}^2(S^2(o,t)\cap E_1)\neq 0 \text{ implies } t \in \left(\|x\|-\frac{r}{\sqrt{1+a}}, \|x\|+\frac{r}{\sqrt{1+a}}\right). \tag{3. 13}$$

In fact, to eliminate the third case in Lemma 3. 5 we just need that

$$(t-\|x\|)^2 < s^2 = r^2 - a(t-\|x\|)^2,$$

and hence

$$|t-\|x\||<\frac{r}{\sqrt{1+a}}.$$

Finally, the second case of Lemma 3.5 does not occur. Indeed, $\|x\| \leqq s-t$ for some $t>0$ would imply

$$(\|x\|+t)^2 \leqq s^2, \quad \text{i.e.} \quad (\|x\|+t)^2 + a(t-\|x\|)^2 \leqq r^2,$$

which gives $\|x\|^2<r^2$, a contradiction to (3. 11). Then we obtain from the Fubini Theorem and Lemma 3. 5 by a lengthy but routine calculation

$$\mathscr{L}^3(E_1) = \int_{\|x\|-\frac{r}{\sqrt{1+a}}}^{\|x\|+\frac{r}{\sqrt{1+a}}} \mathscr{H}^2(S^2(o,t)\cap E_1)\,dt$$

$$= \int_{\|x\|-\frac{r}{\sqrt{1+a}}}^{\|x\|+\frac{r}{\sqrt{1+a}}} \pi\left\{-\frac{1+a}{\|x\|}t^3 + 2(1+a)t^2 - \left[(1+a)\|x\| - \frac{r^2}{\|x\|}\right]t\right\} dt = \frac{4}{3}\frac{\pi r^3}{\sqrt{1+a}}.$$

From (3. 10) we get $\mathcal{H}^3(B^4(m, r)\cap \mathbf{C}_1)=\frac{4}{3}\pi r^3$, which is the LBP. For $\mathbf{C}_2\setminus\{o\}$, the proof is similar.

Proposition 3. 8. *The cone* $\mathbf{C}$ *(or, the manifold* $M=\mathbf{C}\setminus\{o\}$*) satisfies the* GBP *for* $a=1$.

Proof. Assume $a=1$ and $m\in \mathbf{C}_1\setminus\{o\}$.

Let first $r<\sqrt{2}\|x\|$. Then, as in the proof of Proposition 3. 7, we find that

$$\mathcal{H}^3(B^4(m, r)\cap \mathbf{C}_1)=\frac{4}{3}\pi r^3, \quad \mathcal{H}^3(B^4(m, r)\cap \mathbf{C}_2)=0.$$

We only need some more detailed computation to show that $E_2=\emptyset$ and that the integration is still reduced to the first case of Lemma 3. 5.

Let now $r\geqq\sqrt{2}\|x\|$. Then the formula (3. 14) holds again with $a=1$. Here the first case of Lemma 3. 5 occurs for

$$t\in\left(\sqrt{\frac{r^2}{2}-\|x\|^2}, \|x\|+\frac{r}{\sqrt{2}}\right) \quad \text{and the second case for} \quad t\in\left(0, \sqrt{\frac{r^2}{2}-\|x\|^2}\right).$$

We conclude that

$$\mathcal{L}^3(E_1)=\int_0^{\|x\|+\frac{r}{\sqrt{2}}}\mathcal{H}^2(S^2(o, t)\cap E_1)\,dt$$

$$=\int_{\sqrt{\frac{r^2}{2}-\|x\|^2}}^{\|x\|+\frac{r}{\sqrt{2}}}\pi\left\{-\frac{2}{\|x\|}t^3+4t^2-\left(2\|x\|-\frac{r^2}{\|x\|}\right)t\right\}dt+\int_0^{\sqrt{\frac{r^2}{2}-\|x\|^2}}4\pi t^2\,dt.$$

Further, we have the following counterpart of formula (3. 12):

(3. 14) $S^2(o, t)\cap E_2=S^2(o, t)\cap B^3(x, s)$, where $s=\sqrt{r^2-(t+\|x\|)^2}$.

Here the first case of Lemma 3. 5 occurs for $t\in\left(\frac{r}{\sqrt{2}}-\|x\|, \sqrt{\frac{r^2}{2}-\|x\|^2}\right)$ and the second case for $t\in\left(0, \frac{r}{\sqrt{2}}-\|x\|\right)$. We conclude that

$$\mathcal{L}^3(E_2)=\int_0^{\sqrt{\frac{r^2}{2}-\|x\|^2}}\mathcal{H}^2(S(o, t)\cap E_2)\,dt$$

$$=\int_{\frac{r}{\sqrt{2}}-\|x\|}^{\sqrt{\frac{r^2}{2}-\|x\|^2}}\pi\left\{-\frac{2t^3}{\|x\|}-\left(2\|x\|-\frac{r^2}{\|x\|}\right)t\right\}dt+\int_0^{\frac{r}{\sqrt{2}}-\|x\|}4\pi t^2\,dt.$$

Joining both together we get

$$\mathscr{L}^3(E_1)+\mathscr{L}^3(E_2)$$
$$=\int_{\frac{r}{\sqrt{2}}-\|x\|}^{\frac{r}{\sqrt{2}}+\|x\|}\pi\left\{-\frac{2t^3}{\|x\|}-\left(2\|x\|-\frac{r^2}{\|x\|}\right)t\right\}dt+\int_0^{\|x\|+\frac{r}{\sqrt{2}}}4\pi t^2\,dt+\int_0^{\frac{r}{\sqrt{2}}-\|x\|}4\pi t^2\,dt$$
$$=\frac{4}{3}\frac{\pi r^3}{\sqrt{2}}.$$

Hence by Lemma 3. 6

$$\mathscr{H}^3(B^4(m,r)\cap\mathbf{C})=\frac{4}{3}\pi r^3 \quad \text{for all} \quad r>0.$$

We obtain the same formula for $m\in\mathbf{C}_2\setminus\{o\}$. This completes the proof of the GBP for the manifold $\mathbf{C}\setminus\{o\}$. The GBP extends to the origin $o\in M$ by the continuity.

Finally, we have

Proposition 3. 9. *The only cone* $\mathbf{C}$ *from the family* (3. 5) *satisfying the* GBP *is that with* $a=1$. *The half-cone* $\mathbf{C}_1$ (*or* $\mathbf{C}_2$) *cannot satisfy the* GBP *for any value of* a.

Proof. Take $m=o=$ the origin of $\mathbb{R}^4$. Then $x=\pi(m)$ is the origin of $\mathbb{R}^3$ and we get in (3. 9)

$$E_1=E_2=\{z\in\mathbb{R}^3 \mid (1+a)\|z\|^2\leqq r^2\}=B^3\left(o,\frac{r}{\sqrt{1+a}}\right),$$
$$\mathscr{L}^3(E_i)=\frac{4}{3}\pi\left(\frac{r}{\sqrt{1+a}}\right)^3, \quad i=1,2.$$

From (3. 12) and Lemma 3. 6 we obtain

$$\mathscr{H}^3(B^4(o,r)\cap\mathbf{C}_i)=\frac{4}{3}\pi r^3\left(\frac{1}{1+a}\right)<\frac{4}{3}\pi r^3, \quad i=1,2,$$
$$\mathscr{H}^3(B^4(o,r)\cap\mathbf{C})=\frac{4}{3}\pi r^3\left(\frac{2}{1+a}\right).$$

Hence our assertions follow.

We shall close this section by the following "Product Theorem".

Theorem 3. 10. *Let* $M_{n-1}\subset\mathbb{R}^n$ *be a topologically embedded hypersurface of class* C^1. *Then the hypersurface* $M_{n-1}\times\mathbb{R}\subset\mathbb{R}^{n+1}$ *satisfies the* LBP (*or* GBP) *if and only if* M_{n-1} *satisfies the* LBP (*or* GBP, *respectively*).

In fact, we shall prove a more general result:

Theorem 3. 11. *Suppose that G is an open subset of $\mathbb{R}^n$ and M is an m-dimensional topologically embedded C^1-submanifold of G, $m<n$. Then the following two assertions are equivalent:*

(i) $\mathscr{H}^m(B^n(x, r)\cap M)=\alpha_m r^m$

whenever $x\in M$, $r>0$ *and* $B^n(x, r)\subset G$.

(ii) $\mathscr{H}^{m+1}(B^{n+1}(z, r)\cap(M\times\mathbb{R}))=\alpha_{m+1}r^{m+1}$

whenever $z\in M\times\mathbb{R}$, $r>0$ *and* $B^{n+1}(z, r)\subset G\times\mathbb{R}\subset\mathbb{R}^{n+1}$.

Proof. (i) $\Rightarrow$ (ii). If $z=(x, \tau)\in M\times\mathbb{R}$, we use the Fubini Theorem and derive

$$\mathscr{H}^{m+1}(B^{n+1}(z, r)\cap(M\times\mathbb{R}))=\int_{\tau-r}^{\tau+r}\mathscr{H}^m\{B^n(x, \sqrt{r^2-(s-\tau)^2})\cap M\}\,ds$$

$$=\int_{-r}^{r}\alpha(m)(\sqrt{r^2-t^2})^m\,dt=\int_{-r}^{r}\mathscr{L}^m(B^m(o, \sqrt{r^2-t^2}))\,dt=\mathscr{L}^{m+1}(B^{m+1}(o, r))$$

$$=\alpha_{m+1}r^{m+1}.$$

(ii) $\Rightarrow$ (i). Suppose that $x\in M$, $r>0$ and $B^n(x, r)\subset G$. Put $z=(x, \tau)\in M\times\mathbb{R}$. By the Fubini Theorem we get

$$\text{(3. 15)}\quad\begin{cases}\mathscr{H}^{m+1}(B^{n+1}(z, s)\cap(M\times\mathbb{R}))=\displaystyle\int_{M\cap B^n(x,s)}2\sqrt{s^2-\|y-x\|^2}\,d\mathscr{H}^m(y),\\ \mathscr{L}^{m+1}(B^{m+1}(o, s))=\displaystyle\int_{B^m(o,s)}2\sqrt{s^2-\|u\|^2}\,d\mathscr{L}^m(u)\end{cases}$$

for each $s\in(0, r]$.

We claim that

$$\text{(3. 16)}\quad\int_{M\cap B^n(x,s)}(s^2-\|y-x\|^2)^{k+\frac{1}{2}}\,d\mathscr{H}^m(y)=\int_{B^m(o,s)}(s^2-\|u\|^2)^{k+\frac{1}{2}}\,d\mathscr{L}^m(u)$$

for each $s\in(0, r]$ and each $k=0, 1, \ldots$

In fact, if $k=0$, (3. 16) means just the equality of the left-hand sides in (3. 15) and hence it follows from the condition (ii). Assuming that (3. 16) holds for some k, we multiply it by $(2k+3)s$ and integrate with respect to s. Hence

$$\int_0^t(2k+3)\,s\left[\int_{M\cap B^n(x,s)}(s^2-\|y-x\|^2)^{k+\frac{1}{2}}\,d\mathscr{H}^m(y)\right]ds$$

$$=\int_0^t(2k+3)\,s\left[\int_{B^m(o,s)}(s^2-\|u\|^2)^{k+\frac{1}{2}}\,d\mathscr{L}^m(u)\right]ds$$

for every $t\in(0, r]$.

Using the Fubini Theorem, we obtain

$$\int_{M\cap B^n(x,t)} \left[\int_{\|y-x\|}^{t} (2k+3)\, s(s^2-\|y-x\|^2)^{k+\frac{1}{2}}\, ds \right] d\mathscr{H}^m(y)$$
$$= \int_{B^m(o,t)} \left[\int_{\|u\|}^{t} (2k+3)\, s(s^2-\|u\|^2)^{k+\frac{1}{2}}\, ds \right] d\mathscr{L}^m(u).$$

Consequently,

$$\int_{M\cap B^n(x,t)} (t^2-\|y-x\|^2)^{k+\frac{3}{2}}\, d\mathscr{H}^m(y) = \int_{B^m(o,t)} (t^2-\|u\|^2)^{k+\frac{3}{2}}\, d\mathscr{L}^m(u),$$

which is (3. 16) with k replaced by $k+1$.

Since every continuous function on $[0, r^2]$ can be uniformly approximated by polynomials, we infer that

$$\int_{M\cap B^n(x,r)} \varphi(r^2-\|y-x\|^2)\sqrt{r^2-\|y-x\|^2}\, d\mathscr{H}^m(y)$$
$$= \int_{B^m(o,r)} \varphi(r^2-\|u\|^2)\sqrt{r^2-\|u\|^2}\, d\mathscr{L}^m(u)$$

for any continuous function φ on $[0, r^2]$. Thus, if φ_k is a non-decreasing sequence of nonnegative continuous functions on $[0, r^2]$ such that $\varphi_k(t) \to \frac{1}{\sqrt{t}}$ for each $t \in (0, r^2]$, the Monotone Convergence Theorem implies

$$\mathscr{H}^m(M\cap B^n(x, r)) = \alpha_m r^m.$$

This completes the proof.

Remark. The reader familiar with the notion of m rectifiability may notice that it is not necessary to assume that M is embedded. All that is needed in the proof is the equality $\mathscr{H}^{m+1} \llcorner (M\times\mathbb{R}) = (\mathscr{H}^m \llcorner M)\otimes \mathscr{L}^1$ which is true for countably m rectifiable $M\subset\mathbb{R}^n$ (see [3], 2. 10. 45, 2. 10. 46, 3. 2. 23).

4. The quadratic character of the GBP supports

We shall use the following notions and results from the measure theory. (See [3] and [4].)

A subset of $\mathbb{R}^n$ is called a *Borel set* if it belongs to the smallest family $\mathscr{B}$ of subsets of $\mathbb{R}^n$ containing all open sets, and closed under the operations of countable unions and countable intersections. (In particular, $\mathscr{B}$ contains all closed subsets of $\mathbb{R}^n$.) A *function* $f\colon \mathbb{R}^n \to \mathbb{R}$ is said to be *Borel* if the inverse image under f of every open subset of $\mathbb{R}$ is a Borel set of $\mathbb{R}^n$.

A *Borel measure* in $\mathbb{R}^n$ is a map Φ assigning to each Borel set $E \subset \mathbb{R}^n$ a nonnegative number $\Phi(E)$ (possibly ∞) such that

$$\Phi(\emptyset) = 0 \quad \text{and} \quad \Phi\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \Phi(E_i)$$

whenever E_i are disjoint Borel subsets of $\mathbb{R}^n$.

The *m-dimensional Hausdorff measure* in $\mathbb{R}^n$ is a Borel measure defined by the formula

$$(4.1) \quad \mathcal{H}^m(E) = \lim_{\delta \searrow 0} \inf \left\{ \sum_{i=1}^{\infty} \alpha_m 2^{-m} (\operatorname{diam} E_i)^m \mid E \subset \bigcup_{i=1}^{\infty} E_i, \operatorname{diam} E_i < \delta \right\}.$$

(In this definition we get always the same measure if we assume the sets E_i to be arbitrary, Borel, closed, or open, respectively.)

It is well-known that $\mathcal{H}^n = \mathcal{L}^n$ (the Lebesgue measure) on $\mathbb{R}^n$, and that, for any topologically embedded m-dimensional C^1-submanifold M of $\mathbb{R}^n$, $\mathcal{H}^m(M)$ is the usual m-dimensional (Riemannian) volume of M.

If Φ is a Borel measure and E is a Borel set, we define

$$(\Phi \llcorner E)(A) = \Phi(E \cap A).$$

The most important Borel measures for us are those of the form $\Phi = \mathcal{H}^m \llcorner M$, where M is an m-dimensional topologically embedded C^1-submanifold of $\mathbb{R}^n$.

If Φ is a Borel measure over $\mathbb{R}^n$, then the *support* of Φ, spt Φ, is defined by

$$\operatorname{spt} \Phi = \{x \in \mathbb{R}^n \mid \Phi(B^n(x, r) > 0) \quad \text{for every} \quad r > 0\}.$$

It is easy to see that spt Φ is a closed subset of $\mathbb{R}^n$ and that $\Phi(R^n \setminus \operatorname{spt} \Phi) = 0$.

The *integral of a Borel function f with respect to a Borel measure* Φ is defined in a familiar way (see e.g. [4]). Notation: $\int f(x)\, d\Phi(x)$.

A Borel measure Φ in $\mathbb{R}^n$ is said to be *σ-finite* if $\mathbb{R}^n$ can be covered by countably many Borel sets with finite Φ-measure.

The *Fubini Theorem for Borel measures* asserts that, whenever Φ and Ψ are σ-finite Borel measures in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, then there is a unique Borel measure $\Phi \otimes \Psi$ in $\mathbb{R}^n \times \mathbb{R}^m$ such that

$$(\Phi \otimes \Psi)(A \times B) = \Phi(A)\, \Psi(B)$$

holds for all Borel sets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$. Whenever f is a nonnegative Borel function on $\mathbb{R}^n \times \mathbb{R}^m$, then we have

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} f(z)\, d(\Phi \otimes \Psi)(z) = \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^m} f(x, y)\, d\Psi(y) \right] d\Phi(x).$$

We shall often use the following

Corollary of the Fubini Theorem. *Whenever Φ is a σ-finite measure over $\mathbb{R}^n$ and if f is a nonnegative Borel function on $\mathbb{R}^n$, and $E \subset \mathbb{R}^n$ is a Borel set, then*

$$\int_E f(x)\, d\Phi(x) = (\Phi \otimes \mathscr{L}^1)\{(x,t) \in E \times \mathbb{R} \mid 0 \leqq t \leqq f(x)\} = \int_0^{\sup(f)} \Phi\{x \in E \mid f(x) \geqq t\}\, dt \tag{4.2}$$

where $\sup(f)$ *denotes the supremum of f on E.*

Now, we shall prove one of our main results stating that the GBP for a Borel measure implies the "quadratic character" of its support. In what follows, $B(x,r)$ will again denote the $(n+1)$-dimensional open ball in $\mathbb{R}^{n+1}$.

Theorem 4.1. *Suppose that Φ is a nonzero Borel measure in $\mathbb{R}^{n+1}$ such that*

$$\Phi(B(x,r)) = \alpha_n r^n$$

for every $x \in \operatorname{spt}\Phi$ and every $r>0$. Then the support spt Φ *is contained in a (proper) n-dimensional quadric $\mathbf{K} \subset \mathbb{R}^{n+1}$.*

Proof. We can assume that the origin $o \in \operatorname{spt}\Phi$. For every $r>0$ let $b_r \in \mathbb{R}^{n+1}$ be defined by the formula

$$\langle b_r, y\rangle = \frac{n+2}{2\alpha_n r^{n+2}} \int_{B(o,r)} (r^2 - \|z\|^2)\langle z, y\rangle\, d\Phi(z) \tag{4.3}$$

for every $y \in \mathbb{R}^{n+1}$, and let Q_r be a quadratic form on $\mathbb{R}^{n+1}$ defined by the formula

$$Q_r(y) = \frac{n+2}{\alpha_n r^{n+2}} \int_{B(o,r)} \langle z, y\rangle^2\, d\Phi(z) \tag{4.4}$$

for every $y \in \mathbb{R}^{n+1}$.

(Notice that b_r is the center of mass of a "smoothed restriction" of the measure Φ to the ball $B(o,r)$, and $Q_r(y)$ denotes suitably normalized second moment of the restriction of Φ to $B(o,r)$ with respect to the hyperplane $\{y\}^\perp \subset \mathbb{R}^{n+1}$.)

We see easily the following properties of b_r and Q_r:

Proposition 4.2. **(a)** *b_r belongs to the linear span of* spt Φ,

(b) $\|Q_r(y)\| \leqq (n+2)\|y\|^2$ *for every $y \in \mathbb{R}^{n+1}$,*

(c) *there is a sequence $0<r_1<r_2<\cdots$ such that $\lim_{k\to\infty} r_k = \infty$ and such that the sequence of quadratic forms Q_{r_k} converges to a quadratic form Q.*

Proof. **(a)** Suppose that spt $\Phi \subset L$, where $L \subset \mathbb{R}^{n+1}$ is a proper linear subspace. If $y \perp L$, then obviously $\langle b_r, y\rangle = 0$.

(b) It follows from the inequality $\langle z, y\rangle^2 \leqq r^2\|y\|^2$ and the basic assumption about Φ.

(c) For each unit vector $y \in \mathbb{R}^{n+1}$ the family $\{Q_r(y) | r \in \mathbb{R}\}$ is contained in the compact ball $\bar{B}(o, n+2)$ (according to (b)). Further, the quadratic form Q_r is uniquely determined, for each $r \in \mathbb{R}$, by its values $Q_r(y_i)$ on a finite set of unit vectors $\{y_1, \ldots, y_k\}$. Hence the assertion follows.

Applying formula (4. 2) to the σ-finite measure Φ we get

$$\begin{aligned} &\int_{B(o,r)} \|z\|^2 \, d\Phi(z) = (\Phi \otimes \mathscr{L}^1)\{(z, s) \in B(o, r) \times [0, \infty) | 0 \leqq s \leqq \|z\|^2\} \\ &= \int_0^{r^2} \Phi(B(o, r) \setminus B(o, \sqrt{s}))\, ds = \int_0^{r^2} (\alpha_n r^n - \alpha_n (\sqrt{s})^n)\, ds \\ &= \frac{n}{n+2} \alpha_n r^{n+2}, \end{aligned} \tag{4. 5}$$

and hence

$$\text{Trace } Q = \lim_{r \to \infty} \frac{n+2}{\alpha_n r^{n+2}} \int_{B(o,r)} \|z\|^2 \, d\Phi(z) = n. \tag{4. 6}$$

Let us choose now a vector $x \in \operatorname{spt} \Phi$ and a number $r > \|x\|$. Further, let us consider the open balls

$$B_1 = B(x, r - \|x\|), \quad B_2 = B(o, r), \quad B_3 = B(x, r), \quad B_4 = B(x, r + \|x\|),$$

and put

$$J_i = \int_{B_i} (r^2 - \|z - x\|^2)^2 \, d\Phi(z), \quad i = 1, \ldots, 4.$$

Then we have $B_1 \subset B_2 \subset B_4$, $B_1 \subset B_3 \subset B_4$ and hence

$$J_1 \leqq J_2 \leqq J_4, \quad J_1 \leqq J_3 \leqq J_4.$$

We see that

$$J_2 - J_3 \leqq J_4 - J_1, \quad J_3 - J_2 \leqq J_4 - J_1$$

and thus

$$|J_2 - J_3| \leqq J_4 - J_1. \tag{4. 7}$$

Now, for any $z \in B_4 \setminus B_1$ we have

$$r - \|x\| \leqq \|z - x\| \leqq r + \|x\|,$$

which implies

$$-\|x\|^2 - 2r\|x\| \leqq r^2 - \|z - x\|^2 \leqq 2r\|x\| - \|x\|^2,$$

and because $\|x\| \leqq r$, we get

$$|r^2 - \|z - x\|^2| \leqq 3r\|x\|.$$

Hence we obtain, for $\|x\| > 0$,

$$J_4 - J_1 = \int_{B_4 \setminus B_1} (r^2 - \|z-x\|^2)^2 \, d\Phi(z) \leqq 9r^2 \|x\|^2 \, \Phi(B_4 \setminus B_1)$$

$$= 9\alpha_n r^{n+1} \|x\|^3 \frac{\left(1 + \frac{\|x\|}{r}\right)^n - \left(1 - \frac{\|x\|}{r}\right)^n}{\frac{\|x\|}{r}}$$

$$\leqq 9\alpha_n r^{n+1} \|x\|^3 \sup \left\{ \frac{(1+s)^n - (1-s)^n}{s} \,\middle|\, s \in (0, 1) \right\}.$$

Finally, (4. 7) takes on the form

$$|J_2 - J_3| \leqq 9\alpha_n 2^n r^{n+1} \|x\|^3. \tag{4. 8}$$

Here we can derive another expression for J_3 using (4. 2), the GBP assumption, and the fact that $x \in \operatorname{spt} \Phi$ and $o \in \operatorname{spt} \Phi$:

$$J_3 = \int_{B(x,r)} (r^2 - \|z-x\|^2)^2 \, d\Phi(z)$$

$$= (\Phi \otimes \mathscr{L}^1) \{(z, t) \in B(x, r) \times [0, \infty) | t \leqq (r^2 - \|z-x\|^2)^2\}$$

$$= \int_0^{r^4} \Phi(B(x, \sqrt{r^2 - \sqrt{t}})) \, dt = \alpha_n \int_0^{r^4} (\sqrt{r^2 - \sqrt{t}})^n \, dt = \int_0^{r^4} \Phi(B(o, \sqrt{r^2 - \sqrt{t}})) \, dt$$

$$= (\Phi \otimes \mathscr{L}^1) \{(u, t) \in B(o, r) \times [0, \infty) | t \leqq (r^2 - \|u\|^2)^2\}$$

$$= \int_{B(o,r)} (r^2 - \|u\|^2)^2 \, d\Phi(u).$$

(Notice that the resulting equality is formally a transformation of the first integral via the substitution $u = z - x$; yet such a transformation is not a priori allowed in an integral with respect to a measure different from the Lebesgue measure.)

Now, using also the evaluation $\|z-x\|^2 = \|z\|^2 + \|x\|^2 - 2\langle z, x\rangle$ we obtain

$$J_2 - J_3 = \int_{Bo,r)} \{(r^2 - \|z-x\|^2)^2 - (r^2 - \|z\|^2)^2\} \, d\Phi(z)$$

$$= 4 \int_{B(o,r)} (r^2 - \|z\|^2) \langle z, x\rangle \, d\Phi(z) + 4 \int_{B(o,r)} \langle z, x\rangle^2 \, d\Phi(z) \tag{4. 9}$$

$$- 2\|x\|^2 \int_{B(o,r)} (r^2 - \|z\|^2) \, d\Phi(z) + \|x\|^4 \, \Phi(B(o, r))$$

$$- 4\|x\|^2 \int_{B(o,r)} \langle z, x\rangle \, d\Phi(z).$$

Since

$$\|x\|^4 \, \Phi(B(o, r)) = \alpha_n r^n \|x\|^4 \leqq \alpha_n r^{n+1} \|x\|^3,$$

$$\left|4\|x\|^2 \int_{B(o,r)} \langle z, x\rangle \, d\Phi(z)\right| \leqq 4\alpha_n r^{n+1} \|x\|^3,$$

we infer from (4. 8) and (4. 9) that there is a constant $C>0$ such that

$$\Big|4\int_{B(o,r)}(r^2-\|z\|^2)\langle z,x\rangle\,d\Phi(z)+4\int_{B(o,r)}\langle z,x\rangle^2\,d\Phi(z)$$
$$-2\|x\|^2\int_{B(o,r)}(r^2-\|z\|^2)\,d\Phi(z)\Big|\leqq\frac{4\alpha_n}{n+2}\,Cr^{n+1}\|x\|^3.$$

Dividing by $\frac{4\alpha_n}{n+2}\,r^{n+2}=2\int_{B(o,r)}(r^2-\|z\|^2)\,d\Phi(z)$ we obtain

$$|2\langle b_r,x\rangle+Q_r(x)-\|x\|^2|\leqq C\,\frac{\|x\|^3}{r} \tag{4. 10}$$

for every $x\in\operatorname{spt}\Phi$ and every $r>\|x\|$.

From (4. 10) combined with (**c**) of Proposition 4. 2 we immediately see that

$$\lim_{k\to\infty}\langle b_{r_k},x\rangle=\frac{1}{2}(\|x\|^2-Q(x))$$

for every $x\in\operatorname{spt}\Phi$. Using also (**a**) of Proposition 4. 2 we infer that the sequence b_{r_k} converges to some $b\in\mathbb{R}^{n+1}$. Moreover, from (4. 10) we see that

$$\operatorname{spt}\Phi\subset\mathbf{K}=\{z\in\mathbb{R}^{n+1}\,|\,Q(z)-\|z\|^2+2\langle b,z\rangle=0\}\,. \tag{4. 11}$$

Put $Q_1(z)=Q(z)-\|z\|^2$, $z\in\mathbb{R}^{n+1}$. Using (4. 6) we see that Trace $Q_1=-1$, i.e., $Q_1\neq 0$ and $\mathbf{K}\supset\operatorname{spt}\Phi$ is either a proper n-dimensional quadric of $\mathbb{R}^{n+1}$, or (possibly) the one-point set $\{o\}$. In each case, Theorem 4. 1 follows.

Further, we are going to prove

Proposition 4. 3. *Suppose that Φ and $\mathbf{K}$ are as in Theorem* 4. 1, *and that there is a Borel set $E\subset\mathbb{R}^{n+1}$ such that $E\subset\operatorname{spt}\Phi$ and $\Phi=\mathcal{H}^n\llcorner E$. Whenever $M\subset\mathbb{R}^{n+1}$ is a connected n-dimensional smooth submanifold which is contained and relatively open in $\mathbf{K}$, then either $M\subset\operatorname{spt}\Phi$ or $M\cap\operatorname{spt}\Phi=\emptyset$.*

For the proof, we need first a simple geometric lemma.

Lemma 4. 4. *Let M be an n-dimensional submanifold of $\mathbb{R}^{n+1}$ and S an $(n-1)$-dimensional submanifold of M (both topologically embedded and smooth in $\mathbb{R}^{n+1}$). Let $m\in S$ be a fixed point. For every small radius $r>0$, denote by $M_1(r)$ and $M_2(r)$ the two components of the submanifold $B^{n+1}(m,r)\cap(M\setminus S)$. Then*

$$\lim_{r\searrow 0}\frac{\mathcal{H}^n(M_i(r))}{\alpha_n r^n}=\frac{1}{2},\quad i=1,2.$$

Proof. One can choose a local C^1-diffeomorphism F at m which transforms m to o, S to a part of $\mathbb{R}^{n-1} \subset \mathbb{R}^{n+1}$ and M to a part of $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ such that the tangent map $DF(m)$ is an isometry. Then, for $\varepsilon > 0$ given and $r > 0$ sufficiently small, F and F^{-1} satisfy a Lipschitz condition with constant $1+\varepsilon$, and one obtains

$$B(o, (1+\varepsilon)^{-1} r) \subset F(B(m, r)) \subset B(o, (1+\varepsilon) r)$$

and

$$\frac{1}{2}(1+\varepsilon)^{-2n} \leqq (1+\varepsilon)^{-n} \alpha_n^{-1} r^{-n} \mathscr{H}^n(F(M_i(r))) \leqq \alpha_n^{-1} r^{-n} \mathscr{H}^n(M_i(r))$$

$$\leqq (1+\varepsilon)^n \alpha_n^{-1} r^{-n} \mathscr{H}^n(F(M_i(r))) \leqq \frac{1}{2}(1+\varepsilon)^{2n}.$$

Hence we obtain the wanted formula.

Let now $M \subset \mathbb{R}^{n+1}$, $M \subset \mathbf{K}$, be a submanifold as in Proposition 4. 3, and suppose that $M \cap \operatorname{spt} \Phi$ is non-empty. We prove that $M \cap \operatorname{spt} \Phi$ is relatively open in M; because $M \cap \operatorname{spt} \Phi$ is also relatively closed in M and M is connected, this will imply that

$$M \cap \operatorname{spt} \Phi = M.$$

Let us fix $x \in M \cap \operatorname{spt} \Phi$, then we can find $s > 0$ such that, for any $y, z \in B(x, s) \cap M$, $y - z$ is not a normal vector of M at z. We want to show that $M \cap B\left(x, \frac{s}{2}\right) \subset \operatorname{spt} \Phi$. Assuming that there exists $y \in M \cap B\left(x, \frac{s}{2}\right) \setminus \operatorname{spt} \Phi$, we find $z \in \operatorname{spt} \Phi \cap M$ such that $\| y - z \|$ is minimal. Here $z \in B(x, s)$ and thus $y - z$ is not a normal vector of M at z. This means that the sphere $S^n(y, \| y - z \|)$ meets M transversally at z. In other words, the boundary Γ of the intersection $B(y, \| y - z \|) \cap M$ contains z and it is a smooth $(n-1)$-dimensional submanifold of M near z. Moreover,

$$E \cap B(z, r) \subset \operatorname{spt} \Phi \cap B(z, r) \subset B(z, r) \setminus B(y, \| y - z \|) \tag{4. 12}$$

for any $r > 0$ because $B(y, \| y - z \|) \cap \operatorname{spt} \Phi = \emptyset$.

On the other hand, because M is relatively open in $\mathbf{K}$, we get

$$E \cap B(z, r) \subset \mathbf{K} \cap B(z, r) \subset M \tag{4. 13}$$

for any sufficiently small $r > 0$.

Using the LBP, Lemma 4. 4 and (4. 12), (4. 13), we infer that

$$1 = \lim_{r \searrow 0} \frac{\Phi(B(z, r))}{\alpha_n r^n} \leqq \lim_{r \searrow 0} \frac{\mathscr{H}^n(M \cap (B(z, r) \setminus B(y, \| y - z \|)))}{\alpha_n r^n} = \frac{1}{2} \tag{4. 14}$$

which is a contradiction. Hence the assertion of 4. 3 follows.

Theorem 4. 1 and Proposition 4. 3 is all that we need to finish the proof of our main result in its geometric formulation (see the next section). The following Theorem reduces the measure-theoretic formulation to the geometric one. Its proof uses some more complicated tools of the geometric measure theory.

Theorem 4.5. *If a Borel measure Φ in $\mathbb{R}^{n+1}$ satisfies the condition $\Phi(B(x,r))=\alpha_n r^n$ for all $x\in \operatorname{spt}\Phi$ and all $r>0$, then*

$$\Phi=\mathscr{H}^n \llcorner \operatorname{spt}\Phi.$$

Proof. For every Borel set $E\subset\mathbb{R}^{n+1}$ we immediately see that $\Phi(E)\leqq\alpha_n(\operatorname{diam}E)^n$. Hence the definition of the measure $\mathscr{H}^n$ implies that

$$\Phi(E)\leqq 2^n\mathscr{H}^n(E) \quad \text{for every Borel set } E. \tag{4.15}$$

Let $\mathbf{K}$ be the quadric from Theorem 4.1. From [3], 3.2.19, or by a simple direct calculation we see that

$$\lim_{r\searrow 0}\frac{\mathscr{H}^n(B(x,r)\cap\mathbf{K})}{\alpha_n r^n}=1 \quad \text{for } \mathscr{H}^n\text{-almost all } x \text{ in } \mathbf{K}.$$

Since (4.15) and $\operatorname{spt}\Phi\subset\mathbf{K}$ imply that Φ is absolutely continuous with respect to $\mathscr{H}^n\llcorner\mathbf{K}$, we may use the Besicovitch Differentiation Theorem ([3], 2.9.8 and 2.8.18) to conclude that, for the Radon-Nikodym derivative, it holds

$$\frac{d\Phi}{d(\mathscr{H}^n\llcorner\mathbf{K})}(x)=1 \quad \text{at} \quad \mathscr{H}^n\text{-almost every} \quad x\in\operatorname{spt}\Phi,$$

and

$$\frac{d\Phi}{d(\mathscr{H}^n\llcorner\mathbf{K})}(x)=0 \quad \text{at} \quad \mathscr{H}^n\text{-almost every} \quad x\in\mathbf{K}\setminus\operatorname{spt}\Phi.$$

Consequently, $\Phi=\mathscr{H}^n\llcorner\operatorname{spt}\Phi$, q.e.d.

5. Proof of the Main Theorems

Let us start with the observation that, for $n=1$ and $n=2$, our main theorems follow from the results of the previous section and from the local classification (Theorem 3.2). We can exclude at once, as GBP supports, the two-dimensional spheres and any "branched" configuration of half-lines (or half-planes). Thus the corresponding support must be always a unique line, or a unique plane, respectively.

Now, to the general case $n\geqq 3$. Theorem 4.1 suggests that, in order to classify all Borel measures Φ on $\mathbb{R}^{n+1}$ fulfilling the GBP, we have to determine first all quadrics $\mathbf{K}$ of dimension $n\geqq 3$ which satisfy the LBP on some smooth n-dimensional submanifold $M\subset\mathbf{K}$. We shall call these quadrics shortly "LBP-quadrics".

Each proper quadric $\mathbf{K}\subset\mathbb{R}^{n+1}$ can be written in one of the following canonical forms:

$$x_{n+1}^2=\sum_{i=1}^{n}a_i x_i^2+d \quad (a_i,\ d \text{ are constants}), \tag{5.1}$$

$$x_{n+1}=\frac{1}{2}\sum_{i=1}^{n}a_i x_i^2 \quad (a_i \text{ are constants},\ a_1\neq 0). \tag{5.2}$$

Further, we can assume in both cases, without any loss of information, that

$$a_1 a_2 \cdots a_n \neq 0 \quad \text{(the irreducible case).} \tag{5. 3}$$

Indeed, if $a_1 a_2 \cdots a_n = 0$, then the quadric $\mathbf{K}$ is a direct product $\mathbf{K}' \times \mathbb{R}^1$, where $\mathbf{K}'$ is a (proper) quadric of smaller dimension. According to the Product Theorem 3. 10, if $\mathbf{K}$ is an LBP-quadric, then $\mathbf{K}'$ is also an LBP-quadric.

Our next goal is to determine the complete list of irreducible quadrics of dimension $n \geqq 3$ which satisfy, on an open subset, the conditions of the second and the fourth order from Corollary 3. 1. As we shall see, this will practically solve our problem.

We start with the quadrics of "elliptic-hyperbolic" type.

Theorem 5. 1. *For $n \geqq 3$, all n-dimensional irreducible quadrics $\mathbf{K} \subset \mathbb{R}^{n+1}$ of type* (5. 1) *satisfying the condition of the second order* (3. 1) *are described by the following canonical equations:*

(i) $\mathbf{K}$: $x_4^2 = a(x_1^2 + x_2^2 + x_3^2)$, $a > 0$ *arbitrary,*

(ii) $\mathbf{K}$: $x_{n+1}^2 + x_n^2 + \cdots + x_{k+1}^2 = a(x_1^2 + \cdots + x_k^2)$

where $n \geqq 4$, $2 \leqq k \leqq n-1$, and a is a positive solution of the quadratic equation

$$(k-1)(k-3)\, a^2 - 2(k-1)(n-k)\, a + (n-k)(n-k-2) = 0. \tag{5. 4}$$

Proof. We have to fulfil the equation $h^2 = 2\|\vec{h}\|^2$, where $\vec{h}$ is the second fundamental form and h is the mean curvature.

We shall investigate one half of our quadric given by the equation

$$x_{n+1} = \sqrt{D}, \quad \text{where} \quad D = \sum_{i=1}^{n} a_i x_i^2 + d. \tag{5. 5}$$

(The case $x_{n+1} = -\sqrt{D}$ is similar.) We shall limit ourselves to the "generic" points in which $D > 0$.

Using the formula for the metric tensor

$$g_{ij} = \delta_{ij} + \frac{\partial x_{n+1}}{\partial x_i} \cdot \frac{\partial x_{n+1}}{\partial x_j} \qquad (i, j = 1, \ldots, n) \tag{5. 6}$$

we get from (5. 5)

$$g_{ij} = \delta_{ij} + \frac{1}{D} a_i a_j x_i x_j. \tag{5. 7}$$

For the inverse matrix (g^{ij}) we obtain

$$g^{ij} = \delta_{ij} - \frac{1}{D\mathscr{D}} a_i a_j x_i x_j \qquad (i, j = 1, \ldots, n) \tag{5. 8}$$

where

$$\mathscr{D} = \det(g_{ij}) = 1 + \sum_{i=1}^{n} \left(\frac{\partial x_{n+1}}{\partial x_i}\right)^2 = \frac{D + \sum a_i^2 x_i^2}{D}. \tag{5.9}$$

Now, for the second fundamental form $h = (h_{ij})$ we have the formula

$$h_{ij} = \frac{\partial^2 x_{n+1}}{\partial x_i \partial x_j} \cdot \frac{1}{\sqrt{\mathscr{D}}} \quad (i, j = 1, \ldots, n). \tag{5.10}$$

Hence we obtain easily

$$h_{ij} = \frac{1}{\mathscr{E}} (\delta_{ij} a_i D - a_i a_j x_i x_j), \tag{5.11}$$

where $\mathscr{E} = D\sqrt{D\mathscr{D}}$.

Finally, we shall calculate the shape operator $S = (h_i^j)$, given by

$$h_i^j = \sum_{k=1}^{n} g^{jk} h_{ki} \quad \text{for} \quad i, j = 1, \ldots, n. \tag{5.12}$$

We get

$$h_i^j = \frac{1}{\mathscr{F}} \left[\left(\sum_{k=1}^{n} (a_k + 1)\, a_k x_k^2 + d \right) \delta_{ij} a_i - (a_i + 1)\, a_i a_j x_i x_j \right], \tag{5.13}$$

where

$$\mathscr{F} = D\mathscr{D}\sqrt{D\mathscr{D}}. \tag{5.14}$$

Further, using (5.13) we get

$$\mathscr{F} \cdot h = \mathscr{F} \sum_{i=1}^{n} h_i^i = \sum_{i=1}^{n} [a_i(a_i + 1)\,(A - a_i)\, x_i^2] + A d,$$

where

$$A = \sum_{i=1}^{n} a_i, \tag{5.15}$$

and

$$\begin{aligned} \mathscr{F}^2 \cdot \|\vec{h}\|^2 = \mathscr{F}^2 \cdot \sum_{i,j=1}^{n} h_i^j h_j^i &= a_1^2 \left[\sum_{i \neq 1} a_i(a_i + 1)\, x_i^2 + d \right]^2 \\ &+ a_2^2 \left[\sum_{i \neq 2} a_i(a_i + 1) + d \right]^2 + \cdots + a_n^2 \left[\sum_{i \neq n} a_i(a_i + 1)\, x_i^2 + d \right]^2 \\ &+ 2 \sum_{1 \leq i < j \leq n} a_i^2 a_j^2 (a_i + 1)\,(a_j + 1)\, x_i^2 x_j^2. \end{aligned} \tag{5.16}$$

Substituting into the equality $h^2 = 2\|\vec{h}\|^2$ and comparing the coefficients of the resulting 4th order polynomials we get

$$(5.17)\quad \begin{cases} x_i^4\colon & a_i^2(a_i+1)^2[(A-a_i)^2-2(B^2-a_i^2)]=0, \\ x_i^2x_j^2\colon & a_ia_j(a_i+1)(a_j+1)[(A-a_i)(A-a_j)-2a_ia_j-2(B^2-a_i^2-a_j^2)]=0, \\ x_i^2\colon & a_i(a_i+1)[A(A-a_i)-2(B^2-a_i^2)]\,d=0, \\ 1\colon & (A^2-2B^2)\,d^2=0, \end{cases}$$

where

$$(5.18)\qquad B^2=\sum_{i=1}^{n} a_i^2.$$

We make the following observations: Whenever $a_i \neq -1$, then

$$(5.19)\qquad (A-a_i)^2=2(B^2-a_i^2) \quad \text{(from (5. 17)}^1\text{)},$$

$$(5.20)\qquad (A-a_i)\,a_i d=0 \quad \text{(substitute (5. 19) into (5. 17)}^3\text{)},$$

$$(5.21)\qquad (A-2a_i)\,a_i d=0 \quad \text{(substitute (5. 17)}^4 \text{ into (5. 17)}^3\text{)}.$$

Now, we have the following

Proposition 5. 2. *If a_i, $a_j \neq -1$, $i \neq j$, then $a_i = a_j$.*

Proof. Adding (5. 19) for i and j and substituting the resulting expression for B^2 into (5. 17)^2 we obtain $\frac{1}{2}(a_i-a_j)^2=0$.

Proposition 5. 3. *In* (5. 1) *we always have* $d=0$.

Proof. If $a_i=-1$ for all i, then $A=-n$, $B^2=n^2$, and (5. 17)^4 gives $(n-2)nd^2=0$. Because $n \geqq 3$, we get $d=0$. If $a_i \neq -1$ for some i, then (5. 20) and (5. 21) yield again $d=0$.

Proposition 5. 4. *The numbers $a_1, \ldots, a_n$ cannot be all negative.*

Proof. Because $d=0$, the quadric (5. 1) with all negative a_i would have only one real point—the origin.

Summarizing the previous propositions, we get

Proposition 5. 5. *There is an integer k, $1 \leqq k \leqq n$, and a positive real number a such that, after a possible renumeration of the variables $x_1, \ldots, x_n$*

$$(5.22)\qquad a_1=\cdots=a_k=a, \quad a_{k+1}=\cdots=a_n=-1.$$

Proposition 5. 6. *The parameter a in* (5. 22) *satisfies the quadratic equation* (5.4).

Proof. We substitute $A=ka-(n-k)$, $B^2=ka^2+(n-k)$, $a_i=a$ into the equation (5. 19).

Proposition 5. 7. *For $n=3$, the case* (i) *of Theorem* 5. 1 *takes place.*

Proof. If $k=1$, or $k=3$, then the equation (5. 4) is trivially satisfied. For $k=2$, the only solution is $a=-1$. Thus, all 3-dimensional quadrics in question are those with the canonical equation $x_4^2=a(x_1^2+x_2^2+x_3^2)$, $a>0$ arbitrary.

Proposition 5. 8. *For $n\geqq 4$, a positive solution $a>0$ of* (5.4) *may exist only if* $2\leqq k\leqq n-1$.

Proof. For $k=1$, the equation (5. 4) reduces to $(n-1)(n-3)=0$ and for $k=n$ it reduces to $(n-1)(n-3)a^2=0$.

To complete the proof of Theorem 5. 1 we notice that the equation $h^2=2\|\vec{h}\|^2$ is equivalent to (5. 17), which is obviously reduced to the single equation $(A-a)^2=2(B^2-a^2)$. The last condition is now equivalent to the quadratic equation (5. 4) (cf. the proof of Proposition 5. 6).

Now we pass over to the case of the paraboloids.

Theorem 5. 9. *There are no n-dimensional quadrics $\mathbf{K}\subset\mathbb{R}^{n+1}$ of type* (5. 2) *satisfying locally the condition of the second order* (3. 1).

Proof. By similar calculations as in the proof of Theorem 5. 1 we get the analogues of formulas (5. 7) and (5. 8) where we only have to replace the symbol D by 1. Further we get, instead of (5. 11) and (5. 13), the formulas

$$\sqrt{\mathscr{D}}\,h_{ij}=\delta_{ij}a_i \quad (i,j=1,\ldots,n), \tag{5. 23}$$

and

$$\mathscr{D}\sqrt{\mathscr{D}}\,h_i^j=\left(1+\sum_{k=1}^{n}a_k^2x_k^2\right)\delta_{ij}a_i-a_i^2a_jx_ix_j \quad (i,j=1,\ldots,n). \tag{5. 24}$$

Substituting into the equality $h^2=2\|\vec{h}\|^2$ and comparing the coefficients of the resulting 4th order polynomials we get the following analogues of (5. 17):

$$\left\{\begin{array}{ll} x_i^4: & a_i^4[(A-a_i)^2-2(B^2-a_i^2)]=0,\\ x_i^2x_j^2: & a_i^2a_j^2[(Aa_i)(A-a_j)-2a_ia_j-2(B^2-a_i^2-a_j^2)]=0,\\ 1: & A^2-2B^2=0.\end{array}\right. \tag{5. 25}$$

Similarly to Proposition 5. 5 we obtain $a_1=a_2=\cdots=a_n$ and hence (5. 2) takes on the form

$$x_{n+1}=a(x_1^2+\cdots+x_n^2),\ a\neq 0.$$

Substituting into $(5.25)^1$ and $(5.25)^3$ we get

$$(n-1)(n-3)a^2=0,\quad n(n-2)a^2=0,$$

which cannot be satisfied for $a\neq 0$. This completes the proof.

We shall now apply the 4th order condition to the family given by Theorem 5. 1. In the 3-dimensional case, this condition must be automatically satisfied because any quadric of type (i) from Theorem 5. 1 is a LBP quadric (see Proposition 3. 7). Thus the only non-trivial case is when $n \geqq 4$.

Theorem 5. 10. *A cone* $x_{n+1}^2 + x_n^2 + \cdots + x_{k+1}^2 = a(x_1^2 + \cdots + x_k^2)$, $a > 0$, *satisfies the condition of the fourth order,* (3. 2), *if and only if the parameter a fulfils the biquadratic equation*

$$\begin{aligned}&(k-1)(k-2)(k-3)\,a^4 + (k-1)(2k^2-3nk+5n+k-9)\,a^3\\ &+(k-1)(n-k)(3n-13)\,a^2 + (n-k)(2k^2-n^2-kn+7n-5k-6)\,a\\ &+(n-k)(n-k-1)(n-k-2)=0.\end{aligned} \tag{5. 26}$$

Proof. Let us write first our cone in the form

$$x_{n+1}^2 - a x_1^2 = Q \tag{5. 27}$$

where

$$Q = a(x_2^2 + \cdots + x_k^2) - (x_{k+1}^2 + \cdots + x_n^2). \tag{5. 28}$$

Making an orthogonal transformation of the coordinates

$$x'_{n+1} = \frac{x_{n+1} - \sqrt{a}\,x_1}{\sqrt{1+a}}, \quad x'_1 = \frac{\sqrt{a}\,x_{n+1} + x_1}{\sqrt{1+a}}, \quad x'_i = x_i \quad \text{otherwise,}$$

and writing again x_i instead of x'_i we get

$$(1-a)\,x_{n+1}^2 + 2\sqrt{a}\,x_1 x_{n+1} = Q.$$

Finally, making a translation of the origin along the axis $\vec{x}_1$, we obtain

$$(1-a)\,x_{n+1}^2 + 2\sqrt{a}(x_1+\lambda)\,x_{n+1} = Q$$

and for $a \neq 1$ we have

$$x_{n+1} = \frac{-\sqrt{a}(x_1+\lambda) \pm \sqrt{a(x_1+\lambda)^2 + (1-a)\,Q}}{1-a}. \tag{5. 29}$$

(The case $a = 1$ will be settled later.) We shall choose the plus sign in (5. 29) (the upper half of the cone). For $\lambda \neq 0$ we can take the power series expansion in (5. 31) and we get easily

$$x_{n+1} = Z_2 + Z_3 + Z_4 + \cdots \tag{5. 30}$$

where

$$Z_2 = \frac{Q}{2\lambda\sqrt{a}}, \quad Z_3 = -\frac{Q x_1}{2\lambda^2\sqrt{a}}, \quad Z_4 = \frac{Q x_1^2}{2\lambda^3\sqrt{a}} - \frac{Q^2}{8\lambda^3 ab} \tag{5. 31}$$

and $b = \dfrac{\sqrt{a}}{1-a}$.

Comparing (5. 30) with (2. 36) and (2. 37) we get

$$\lambda_2 = \cdots = \lambda_k = \frac{\sqrt{a}}{\lambda}, \quad \lambda_{k+1} = \cdots = \lambda_n = -\frac{1}{\lambda\sqrt{a}}, \quad \lambda_1 = 0, \tag{5. 32}$$

$$\begin{cases} C_{1ii} = -\dfrac{\sqrt{a}}{\lambda^2} \quad (i = 2, \ldots, k), \quad C_{1\alpha\alpha} = \dfrac{1}{\lambda^2\sqrt{a}} \quad (\alpha = k+1, \ldots, n), \\ C_{uvw} = 0 \quad \text{for other combinations of indices} \\ \qquad\qquad \text{(apart from symmetry)}, \end{cases} \tag{5. 33}$$

$$\begin{cases} D_{11ii} = \dfrac{2\sqrt{a}}{\lambda^3}, \quad D_{11\alpha\alpha} = -\dfrac{2}{\lambda^3\sqrt{a}}, \quad \begin{pmatrix} i, j = 2, \ldots, k \\ \alpha, \beta = k+1, \ldots, n \end{pmatrix} \\ D_{iiii} = \dfrac{-3a^2}{\lambda^3 ab}, \quad D_{iijj} = \dfrac{-a^2}{\lambda^3 ab}, \\ D_{\alpha\alpha\alpha\alpha} = \dfrac{-3}{\lambda^3 ab}, \quad D_{\alpha\alpha\beta\beta} = \dfrac{-1}{\lambda^3 ab}, \\ D_{ii\alpha\alpha} = \dfrac{a}{\lambda^3 ab}, \quad D_{uvwz} = 0 \quad \text{for other combinations of indices.} \end{cases} \tag{5. 34}$$

From (5. 32) and (2. 19) we get

$$h = \frac{(k-1)\,a - (n-k)}{\lambda\sqrt{a}}, \quad \|\vec{h}\|^2 = \frac{(k-1)\,a^2 + (n-k)}{\lambda^2 a} \tag{5. 35}$$

and from (5. 33) and (2. 26) we obtain

$$\begin{cases} \|\nabla\vec{h}\|^2 = \sum_{i,j,k=1}^{n} (C_{ijk})^2 = \dfrac{3}{\lambda^4 a}\,[(k-1)\,a^2 + (n-k)], \\ \|d\,h\|^2 = \sum_{i=1}^{n} (\nabla_i h)^2 = (\nabla_1 h)^2 = \dfrac{(k-1)^2\,a^2 - 2(k-1)\,(n-k)\,a + (n-k)^2}{\lambda^4 a}. \end{cases} \tag{5. 36}$$

Further, it follows from (2. 29), (5. 32), (5. 34) and (5. 35) that

$$\Delta h = \frac{1}{\lambda^3\sqrt{a}}\,(n-3)\,[(n-k) - (k-1)\,a]. \tag{5. 37}$$

Finally, using (2. 20), (2. 21) and (5. 34), (5. 37) we obtain

$$\|R\|^2 = \frac{2[(k-1)(k-2)a^4 + 2(k-1)(n-k)a^2 + (n-k)(n-k-1)]}{\lambda^4 a^2} \tag{5. 38}$$

$$\begin{aligned}\|\varrho\|^2 = \frac{1}{\lambda^4 a^2}\{&(k-1)(k-2)^2 a^4 - 2(n-k)(k-1)(k-2)a^3 \\ &+ (k-1)(n-k)(n-1)a^2 - 2(n-k)(k-1)(n-k-1)a \\ &+ (n-k)(n-k-1)^2\}.\end{aligned} \tag{5. 39}$$

Substituting from (5. 35)—(5. 39) into (3. 2) we obtain, after a lengthy but routine calculation, the required formula (5.26).

We can check directly that the same formula applies also to the case $a = 1$. This completes the proof of Theorem 5. 10.

The rest of our calculations will be a pure algebra.

Theorem 5. 11. *In dimensions $n \geqq 4$ there are no irreducible quadrics satisfying locally both conditions* (3. 1) *and* (3. 2).

Proof. A cone $x_{n+1}^2 + x_n^2 + \cdots + x_{k+1}^2 = a(x_1^2 + \cdots + x_k^2)$, $a > 0$ satisfies the geometric conditions (3. 1), (3. 2) if and only if the parameter a satisfies the corresponding algebraic equations (5. 4) and (5. 26). Let us make a substitution $p = k-1$, $q = n-k$. Then these algebraic equations can be written in the form

$$h_2(x) = p(p-2)x^2 - 2pqx + q(q-2) = 0, \tag{5. 40}$$

$$h_4(x) = p(p-1)(p-2)x^4 + pAx^3 + pqCx^2 + qBx + q(q-1)(q-2) = 0, \tag{5. 41}$$

where

$$\begin{aligned}&A = -3pq - p^2 + 4p + 2q - 4, \quad B = -3pq - q^2 + 4q + 2p - 4, \\ &C = 3p + 3q - 10.\end{aligned} \tag{5. 42}$$

Any common positive solution of the equations (5. 40) and (5. 41) would be also a root of the polynomial

$$\begin{aligned}\tilde{h}_2(x) &= (2-p-q)h_2(x) - \frac{1}{x}[h_4(x) - ((p-1)x^2 + (q-1))h_2(x)] \\ &= 2pqx^2 + Dx + 2pq,\end{aligned}$$

where $D = 2pq - p^2 - q^2 + 2p + 2q$.

Recall the formula for the *resultant* of two quadratic polynomials $a_0x^2+a_1x+a_2$, $b_0x^2+b_1x+b_2$:

$$R=(a_0b_2-b_0a_2)^2-(a_0b_1-b_0a_1)(a_1b_2-b_1a_2).$$

Then the resultant of the polynomials $h_2(x)$, $\tilde{h}_2(x)$ is given by

$$\begin{aligned}R&=4p^2q^2(p(p-2)-q(q-2))^2+(p(p-2)D+4p^2q^2)(q(q-2)+4p^2q^2)\\&=p^2q^2(2p(p-2)+2q(q-2)+D)^2+(pq(p-2)(q-2)-p^2q^2)(D^2-16p^2q^2)\\&=p^2q^2(p+q)^2(p+q-2)^2+2pq(p+q-2)(4pq-D)(4pq+D)\\&=pq(p+q)(p+q-2)^2[p^2(q-2)+q^2(p-2)+12pq+4p+4q].\end{aligned}$$

Hence, if $p\geqq 2$ and $q\geqq 2$, the resultant is positive.

Because R is symmetric with respect to p, q, and because we can assume $p, q\geqq 1$ (cf. Proposition 5. 8) and $p+q=n-1\geqq 3$, it remains to consider the case $p=1$ and $q\geqq 2$. But then the equation $R=0$ reduces to

$$q^2-17q-2=0,$$

which does not have integer roots.

Hence $R\neq 0$, which shows that (5. 40) and (5. 41) do not have common positive solutions. This concludes the proof of Theorem 5. 11.

Remark 5. 12. As we see from Theorem 5. 1, (ii), and Theorem 5. 11, there are many examples of n-dimensional submanifolds $M\subset\mathbb{R}^{n+1}$ which satisfy the condition of the second order for the LBP but do not fulfil the LBP.

We can now finish the proof of our Main Theorems. We have the following consequence of Theorem 5. 11, Theorem 3. 2, Proposition 3. 7 and Theorem 3. 10:

Corollary 5. 13. *The only n-dimensional* LBP *quadrics* $\mathbf{K}\subset\mathbb{R}^{n+1}$ *are those which are congruent to some of the following direct products*:

(a) $P\times\mathbb{R}^{n-1}$, *where* P *is a couple of lines in* $\mathbb{R}^2$ *given by*

$$x_1^2=c\quad(c\geqq 0),\quad \text{or}\quad bx_1^2-x_2^2=0\quad(b>0).$$

(b) $S^2(r)\times\mathbb{R}^{n-2}$, *where* $S^2(r)$ *is the sphere* $x_1^2+x_2^2+x_3^2=r^2$,

(c) $\mathbf{C}\times\mathbb{R}^{n-3}$, *where* $\mathbf{C}$ *is a cone* $x_4^2=a(x_1^2+x_2^2+x_3^2)$, $a>0$.

Consider now the components M_1, M_2 *of* $M=\mathbf{C}\setminus\{o\}$, which are connected smooth submanifolds of $\mathbf{C}$, and put $\mathbf{C}_i=\bar{M}_i$, $i=1, 2$ (cf. formula (3. 10)). According to Proposition 4. 3 and Theorem 4. 5, if $\mathbf{K}=\mathbf{C}\times\mathbb{R}^{n-3}$ and $\operatorname{spt}\Phi\subset\mathbf{K}$ for some Borel measure Φ satisfying the GBP (Theorem 4. 1), then we must have one of the following cases: $\operatorname{spt}\Phi=\mathbf{K}$, or $\operatorname{spt}\Phi=\mathbf{C}_i\times\mathbb{R}^{n-3}$ for some of the half-cones $\mathbf{C}_1$, $\mathbf{C}_2$. According to Proposition 3. 9 and the Product Theorem 3. 10, the "half-cone" $\mathbf{C}_i\times\mathbb{R}^{n-3}$ never satisfies the GBP, and the "cone" $\mathbf{C}\times\mathbb{R}^{n-3}$ will satisfy the GBP only for $a=1$.

Further, if $\operatorname{spt}\Phi\subset\mathbf{K}=S^2(r)\times\mathbb{R}^{n-2}$, then $\mathbf{K}$ is a connected smooth manifold and necessarily $\operatorname{spt}\Phi=\mathbf{K}$. Because $S^2(r)$ does not satisfy the GBP, the quadric $\mathbf{K}$ cannot satisfy the GBP, as well.

Finally, we can discuss easily the various situations for a couple of lines, P, to see that the only case when the GBP holds is when P is the (double) line $(x_1)^2 = 0$ and thus $\operatorname{spt} \Phi = \mathbb{R} \times \mathbb{R}^{n-1} = \mathbb{R}^n$. This completes the proof of the measure-theoretic version of our Main Theorem.

As concerns the geometric version, we only notice that if $\Phi = \mathscr{H}^n \llcorner M$ for a C^1-submanifold $M \subset \mathbb{R}^{n+1}$, then $\operatorname{spt} \Phi$ is the topological closure $\bar{M}$ of M in $\mathbb{R}^{n+1}$. Hence M is open and dense in the corresponding supporting quadric $\mathbf{K}$. Because $\mathbf{K}$ satisfies the GBP and so does M, we conclude that $\mathscr{H}^n(\mathbf{K} \setminus M) = 0$.

We shall end up with a conjecture and a problem.

Conjecture 5. 14. *Each connected topologically embedded smooth hypersurface $M_n \subset \mathbb{R}^{n+1}$ satisfying the* LBP *is a part of an n-dimensional quadric.*

Problem 5. 15. *Look for new examples of* GBP (*and* LBP) *submanifolds in larger codimensions.*

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