Math 583

Homework due May 12

Part 1

1 Cacciopoli Inequality

Definition 1 *Assume that $f \in L^1_{\text{loc}}(\Omega)$, $1 \leq i \leq n$. We say that $g_i \in L^1_{\text{loc}}(\Omega)$ is the weak partial derivative of f with respect to x_i in Ω if*

$$\int_\Omega f \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega g_i \varphi dx \tag{1}$$

for all $\varphi \in C^1_c(\Omega)$. Notation $g_i = \frac{\partial f}{\partial x_i}$ and $\nabla f = \left(\frac{\partial f}{\partial x_i}, \cdots, \frac{\partial f}{\partial x_n}\right)$.

Definition 2 *The function f is said to be a Sobolev function if $f \in W^{1,p}_{\text{loc}}(\Omega)$ for some $1 \leq p < \infty$. Here*

$$W^{1,p}_{\text{loc}}(\Omega) = \{f : \Omega \to \mathbb{R} : f, \nabla f \in L^p_{\text{loc}}(\Omega)\} \tag{2}$$

and

$$W^{1,p}(\Omega) = \{f : \Omega \to \mathbb{R} : f, \nabla f \in L^p(\Omega)\}. \tag{3}$$

Furthermore if $f \in W^{1,p}(\Omega)$

$$\|f\|_{W^{1,p}(\Omega)} = \left(\int_\Omega (|f|^p + |\nabla f|^p) dx\right)^{\frac{1}{p}}. \tag{4}$$

Definition 3 *Let $u \in W^{1,2}_{\text{loc}}(\Omega) \cup C(\Omega)$; u is said to be weakly harmonic if*

$$\int_\Omega \langle \nabla \varphi, \nabla u \rangle dx = 0 \tag{5}$$

for all $\varphi \in C^\infty_c(\Omega)$.

Theorem 1 (Weyl's lemma) *If $u \in W^{1,2}_{\text{loc}}(\Omega) \cup C(\Omega)$ is weakly harmonic then $u \in C^\infty(\Omega)$ and $\Delta u = 0$.*

Remark 1 Since (5) holds for all $\varphi \in C^\infty(\Omega)$ then it also holds for all for all $\psi \in W_0^{1,2}(\Omega)$ where $W_0^{1,2}(\Omega)$ denotes the completion of $C^\infty(\Omega)$ in $W^{1,2}(\Omega)$.

Problem 1 (Cacciopoli Inequality) *Let u be a harmonic function on a regular domain $\Omega \subset \mathbb{R}^n$ satisfying $u = 0$ on $\partial\Omega \cap B(Q, 2r)$ where $Q \in \partial\Omega$. Then for $s \in (0, r)$*

$$\int_{B(Q,s)\cap\Omega} |\nabla u|^2 \leq \frac{C}{s^2} \int_{B(Q,2s)} u^2.$$

Hint: *For any $\varphi \in C_c^\infty(B(Q, 2s))$, $u\varphi^2 \in W_0^{1,2}(\Omega)$. Choose φ appropriately and apply the remark above.*

2 Derivative estimates for harmonic functions in $\mathbb{R}^n$

Let v be a harmonic function in $\mathbb{R}^n$, using the Poisson integral formula, we have that for $r > 0$

$$v(X) = \frac{r^2 - |X|^2}{\sigma_{n-1} r} \int_{\partial B(0,r)} \frac{v(\zeta)}{|X - \zeta|^n} d\zeta \tag{6}$$

where σ_{n-1} is the area of the unit sphere in in $\mathbb{R}^n$.

Problem 2 *Let v be a harmonic function in $\mathbb{R}^n$, prove that for $X \in B(0, \frac{r}{2})$, $r > 1$, $k \geq 1$ and any multi-index of length k, $(\alpha_1, \cdots, \alpha_k)$,*

$$|\partial_{\alpha_1} \cdots \partial_{\alpha_k} v(X)| \leq \frac{C}{r^k} \max_{\zeta \in \partial B(0,r)} |v(\zeta)|. \tag{7}$$

3 Green function with pole at infinity

Definition 4 *Let $M > 1$. The ball $B(X, r) \subset \Omega$ is said to be M-non-tangential in Ω if*

$$M^{-1} r \leq d(B(X, r), \partial\Omega) \leq Mr. \tag{8}$$

Definition 5 *A bounded domain $\Omega \subset \mathbb{R}^n$ is called non-tangentially accessible (NTA) if there exists constants $M > 2$ and $R > 0$ such that*

1. *Ω satisfies the corkscrew condition: $\forall\, Q \in \partial\Omega$, $\forall\, r \in (0, R)$ there exists $A = A(Q, r) \in \Omega$ such that*
$$\frac{r}{M} \leq |A - Q| \leq r \text{ and } d(A, \partial\Omega) \geq \frac{r}{M}$$

2. Ω^C *satisfies the corkscrew condition.*

3. Ω *satisfies the Harnack Chain Condition; if* $\epsilon > 0$, $r \in (0, R)$ *and* $X_1, X_2 \in B(Q, \frac{r}{4}) \cap \Omega$ *with* $|X_1 - X_2| \leq 2^k \epsilon$ *and* $d(X_i, \partial\Omega) \geq \epsilon$ *for* $i = 1, 2$, *there exists a chain of* Mk *balls* $B_1, \ldots, B_{Mk}$ *in* Ω *connecting* X_1 *to* X_2 ($X_1 \in B_1, X_2 \in B_{Mk}$) *satisfying*

$$\begin{aligned} \text{diam}B_j &\sim d(B_j, \partial\Omega) \\ \text{diam}B_j &\geq C^{-1} \min\{d(X_1, B_j), d(X_2, B_j)\} \end{aligned}$$

for some $C > 1$.

Condition 3 guarantees that the Harnack principle for non-negative harmonic functions holds in Ω, namely if $\Delta u = 0$ in Ω, $u \geq 0$ in Ω and $X_1, X_2 \in Br(\frac{r}{4}, Q) \cap \Omega$ as above then $M^{-k} u(X_1) \leq u(X_2) \leq M^k u(X_1)$. Morally speaking this inequality (i.e. Harnack's principle) insures that all values of u in Ω are comparable, and that the constants involved only depend on the NTA constants and the relative distance of the points to the boundary.

Definition 6 *An unbounded domain* $\Omega \subset \mathbb{R}^n$ *is said to be non-tangentially accessible (NTA) if the previous definition holds for* $R = +\infty$.

Theorem 2 *Let* $\Omega \subset \mathbb{R}^n$, $n \geq 2$ *be an NTA domain. Then* Ω *is regular for the Dirichlet problem.*

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be an unbounded NTA domain. Let $X \in \Omega$, and $G(X, -)$ denote the Green's function of Ω with pole at X. Note that if $X \in \Omega \backslash B(Q, 2R)$ where $Q \in \partial\Omega$ and $R > 0$ then

$$\left\{ \begin{array}{rl} \Delta G(X, -) = 0 & \text{in } \Omega \cap B(Q, R) \\ G(X, -) = 0 & \text{on } \partial\Omega \cap B(Q, R) \\ G(X, -) > 0 & \text{in } \Omega \cap B(Q, R) \end{array} \right. \tag{9}$$

Furthermore if $\varphi \in C_c^\infty(B(Q, R))$ then

$$\int_\Omega G(X, Y) \Delta\varphi(Y)\, dY = -\int_\Omega \nabla G(X, Y) \cdot \nabla\varphi(Y)\, dY = \int_{\partial\Omega} \varphi(Q) d\omega^X(Q). \tag{10}$$

Recall the following results about NTA domains

Lemma 1 ([JK], Lemma 4.1) *Let* Ω *be an NTA domain, let* $K \subset \mathbb{R}^n$ *be a compact set. There exists* $\beta > 0$ *such that for all* $Q \in \partial\Omega \cap K$, $0 < 4r < R$, *and every harmonic function* u *in* $\Omega \cap B(Q, 2r)$, *if* u *vanishes continuously on* $B(Q, 4r) \cap \partial\Omega$, *then for* $X \in \Omega \cap B(Q, r)$,

$$|u(X)| \leq C \left(\frac{|X - Q|}{r} \right)^\beta sup\{|u(Y)| : Y \in B(Q, 2r) \cap \Omega\},$$

where C *depends only on* K *and on the NTA constants.*

Lemma 2 ([JK], Lemma 4.4) *Let Ω be an NTA domain, let $K \subset \mathbb{R}^n$ be a compact set, $Q \in \partial\Omega \cap K$, and $0 < 4r < R$. If $u \geq 0$ is a harmonic function in Ω, and u vanishes continuously on $B(Q, 2r)$, then*

$$u(Y) \leq Cu(A(Q, r))$$

for all $Y \in B(Q, r) \cap \Omega$. Here C depends only on K and on the NTA constants.

Lemma 3 ([JK], Lemma 4.8) *Let Ω be an NTA domain, let $K \subset \mathbb{R}^n$ be a compact set, $Q \in \partial\Omega \cap K$, $0 < 2r < R$, and $X \in \Omega \backslash B(Q, 2r)$. Then*

$$C^{-1} < \frac{\omega^X(B(Q, r))}{r^{n-2}G(A(Q, r), X)} < C,$$

where $G(A(Q, r), -)$ is the Green's function of Ω with pole at $A(Q, r)$.

Lemma 4 ([JK], Lemma 4.9, 4.11) *Let $\Omega \subset \mathbb{R}^n$ be an NTA domain with constants $M > 1$ and $R > 0$, let $K \subset \mathbb{R}^n$ be a compact set, $Q \in \partial\Omega \cap K$, $0 < 2r < R$, and $X \in \Omega \backslash B(Q, 2Mr)$. Then for $s \in [0, r]$*

$$\omega^X(B(Q, 2s)) \leq C\omega^X(B(Q, s)) \tag{11}$$

where C depends only on the NTA constants of Ω and on K.

Problem 3 *Let $\Omega \subset \mathbb{R}^n$ be an unbounded NTA domain. Show that there exists a function u such that*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{12}$$

Hint: *For $X \in \Omega$, let $G(X, -)$ denote the Green's function of Ω with pole at X. Let $Q \in \partial\Omega$ and consider the function*

$$u_X(-) = \frac{G(X, -)}{G(X, A)},$$

where $A = A(Q, 1)$. Consider a sequence $u_j = u_{X_j}$ where $\{X_j\}_j$ is a sequence of points in Ω such that $X_j \to \infty$.

Note: *This function is unique up to multiplication by a positive constant (you are not asked to prove this).*

References

[JK] D. Jerison & C. Kenig, Boundary Behavior of Harmonic Functions in Nontangentially Accessible Domains, *Adv. in Math.* **46** (1982), 80–147.