Math 525

Homework due 02/05/03

Reading:

Sections 1, 2, 3 and 4, Chapter 8 from Royden. Sections 1 and 2, Chapter 4 from Folland.

Problems from Folland:

Chapter 3, Section 4: problem 25.

Problem 1. Show that there exists a constant $C > 0$ such that for every $f \in L^1_{loc}(\mathbb{R}^n)$

$$m\left(\{x \in \mathbb{R}^n : Mf(x) > \alpha\}\right) \leq \frac{C}{\alpha} \int_{\{x:\ |f(x)| \geq \frac{\alpha}{2}\}} |f(y)|\, dy.$$

Recall that for $x \in \mathbb{R}^n$

$$Mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\, d(y).$$

Problem 2. Let $p \geq 1$, and let $f : \mathbb{R}^n \to [0, \infty)$ be measurable. Let

$$A_t = \{x \in \mathbb{R}^n : f(x) > t\}.$$

Show that

$$\int_{A_0 \times (0,\infty)} t^{p-1} \chi_{A_t}(x)\, dx\, dt = \int_0^\infty t^{p-1} m(A_t)\, dt = \frac{1}{p} \int_{A_0} f^p(x)\, dx$$

Problem 3. For $p > 1$, define

$$L^p(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R} : \int_{\mathbb{R}^n} |f(y)|^p \, dy < \infty \right\}.$$

Show that there exists a constant $C(p) > 1$ depending on p such that

$$\left(\int_{\mathbb{R}^n} [Mf(x)]^p \, dx \right)^{\frac{1}{p}} \leq C(p) \left(\int_{\mathbb{R}^n} |f|(x)^p \, dx \right)^{\frac{1}{p}},$$

for all $f \in L^p(\mathbb{R}^n)$. (Hint: Use problems 1 and 2.)

Problem 4. Let L^n denote Lebesgue measure on $\mathbb{R}^n$ and let μ be a Borel regular measure on $\mathbb{R}^n$. Assume that for all $x \in \mathbb{R}^n$

$$\limsup_{r \to 0} \frac{\mu(B(x,r))}{\omega_n r^n} \geq 1,$$

where ω_n denotes the Lebesgue measure of the ball of center 0 and radius 1 in $\mathbb{R}^n$. Prove that for any Borel subset $A \subset \mathbb{R}^n$

$$\mu(A) \geq L^n(A).$$