Math 525

Homework due 01/29/03

Reading: Chapter 5 of Royden. Sections 5 and 6, Chapter 11 form Royden. Sections 1, 2, 4 and 5, Chapter 3 from Folland.

Problems from Folland:

Chapter 3, Section 1 : problems 2, 3.

Chapter 3, Section 2 : problems 8, 11.

Problem 1. For $f \in L^1_{loc}(\mathbb{R}^n)$, and $0 < s < n$ define

$$\Lambda_s = \{x \in \mathbb{R}^n : \liminf_{r \to 0} \frac{1}{r^s} \int_{B(x,r)} |f|(y)\, dy > 0.\}$$

Prove that $m(\Lambda_s) = 0$, for all $s \in (0, n)$.

$*$**Problem.** A positive measure μ on $\mathbb{R}^n$ is said to be a doubling measure if there exist $C_0 > 1$ such that

$$\mu(B(x, 2r)) \leq C_0 \mu(B(x, r)), \quad \forall x \in \mathbb{R}^n \text{ and } \forall r > 0.$$

For $f \in L^1_{loc}(\mu)$ define the Hardy-Littlewood maximal function Mf by

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\, d\mu(y).$$

Let μ be a doubling measure on $\mathbb{R}^n$. Show that there exists a constant $C > 0$ such that for all $f \in L^1(\mu)$ and all $\alpha > 0$,

$$\mu\left(\{x \in \mathbb{R}^n : Mf(x) > \alpha\}\right) \leq \frac{C}{\alpha} \int_{\mathbb{R}^n} |f(y)|\, d\mu(y).$$