Math 524: Real Analysis
Final Exam, Fall 2002

Tatiana Toro, Instructor

Due: Friday December 13, 2002, 2pm in Padelford C-332

- Do each of the 5 problems below.
- Do each problem on a separate sheet of paper. Stable your solutions together, in numerical order, before handing them in. Solutions will be graded for clarity, completeness and rigor.
- You may consult the two text books *Real Analysis* by G.R. Folland and *Real Analysis* by H.L. Royden, your class notes and your returned homework assignments for the course.
- If you refer to one of the texts, give precise citations. If you use a problem which was not assigned, or a $(*)$ problem you **must** give a complete solution of it as part of your answer.
- You may **not** consult any other sources. In particular, please do not look at any other textbooks or discuss this exam with your colleagues or anyone else before Friday at 2:00 PM.

1. Let X be a compact metric space and μ a finite, nonnegative Borel measure on X. Suppose that $\mu\{x\} = 0$ for every $x \in X$. Prove that for every $\epsilon > 0$, there is $\delta > 0$ such that
$$\mu(E) < \epsilon,$$
whenever E is a Borel set in X having diameter $\leq \delta$.

2. Give either a proof or a counterexample for each of the following statements:

   - Let X be a compact separable metric space then $C(X)$ is complete.
   - Let X be a compact separable metric space then $C(X)$ is compact.
   - Let μ be a σ-finite complete measure on a space Y, then $L^1(\mu)$ is complete.
   - Let μ be a σ-finite complete measure on a space Y, then $L^1(\mu)$ is compact.

3. For every $\epsilon \in (0,1)$, show that there exists a set $E \subset [0,1]$ such that

$$m(E^c) \geq 1 - \epsilon$$

   and

$$m(E \cap [a,b]) > 0$$

   for every interval $[a,b] \subset [0,1]$ with $0 < a < b < 1$.

4. Let $k \in C([0,1] \times [0,1])$ and assume that $|k(x,y)| < 1$ for $(x,y) \in [0,1] \times [0,1]$. Let $\varphi \in C([0,1])$. Prove that there exits a unique function $f \in C([0,1])$ such that

$$f(x) = \varphi(x) + \int_0^1 k(x,y) f(y)\, dy.$$

   Hint: Recall Banach's fixed point theorem and its proof. If you apply this theorem show that all the hypothesis are satisfied.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume $\mu(X) < \infty$. Let $\{f_n\}_{n\geq 1}$ be a sequence in $L^1(\mu)$ satisfying:

   (a) $f_n \to f$ $\mu - a.e$ in X

   (b) Given $\epsilon > 0$ there is $\delta > 0$ so that

$$\sup_{n\geq 1} \int_E |f_n|\, d\mu < \epsilon, \quad \text{whenever} \quad \mu(E) < \delta.$$

   Show that $f \in L^1(\mu)$ and that $f_n \to f$ in $L^1(\mu)$, i.e.

$$\lim_{n\to\infty} \int |f_n - f|\, d\mu = 0.$$