Math 524

Homework due 11/13/02

Problem 1. Le X be a metric space, let $\mathcal{B}$ be the Borel σ-algebra over X. A measure μ is said to be **outer regular** if for every Borel set E

$$\mu E = \inf \{\mu O : E \subset O, O \text{ open}\}.$$

It is said to be **inner regular** if

$$\mu E = \sup \{\mu K : K \subset E, K \text{ compact}\}.$$

The measure μ is **Borel regular** if it is both inner and outer regular.

Let μ be a σ-finite Borel regular measure on X.

1.1 Show that for every Borel set E and every $\epsilon > 0$ there exists an open set O such that $E \subset O$ and

$$\mu(O \backslash E) < \epsilon$$

1.2 Show that for every Borel set E, such that $\mu(E) < \infty$, and every $\epsilon > 0$ there exists a compact set K such that $K \subset E$ and

$$\mu(E \backslash K) < \epsilon$$

Problem 2. Let (X, ρ) and (Y, σ) be metric spaces. Let $f : X \to Y$. For each $x \in X$, define

$$\omega(x) = \inf_{r>0} \{\operatorname{diam}(f(B(x, r)))\}.$$

For $E \subset Y$, $\operatorname{diam} E = \sup_{x,y \in E} \{\sigma(x, y)\}$. The function ω is called the *oscillation function* for f. Prove the following statements.

2.1. The function f is continuous at x if and only if $\omega(x) = 0$.

2.2. For each $\alpha \in \mathbb{R}$, the set $\{x \in X : \omega(x) < \alpha\}$ is open in X.

2.3. The set $\{x \in X : f \text{ is continuous at } x\}$ is a G_δ set, i.e. it can be written as a countable intersection of open sets.

2.4. There is no real valued function f defined on $\mathbb{R}$ such that
$\{x \in \mathbb{R} : f \text{ is continuous at } x\} = \mathbb{Q}$.