Math 524

Homework due 10/23/02

Problem 1. Let (X, ρ) be a metric space. Let $E \subset X$.

1.1 Prove that every uniformly convergent sequence of bounded functions on E is uniformly bounded on E.

1.2 Assume that $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions which converge uniformly on E. Prove that $\{f_n g_n\}$ converges uniformly on E.

1.3 Assume that $\{f_n\}$ and $\{g_n\}$ converge uniformly on E. Does $\{f_n g_n\}$ converge uniformly on E?

Problem 2. Let (X, ρ) be a metric space. Let $K \subset X$ be compact. For each $k \in \mathbb{N}$, let $f_k : K \to \mathbb{R}$ be continuous and satisfy:
(i) $f_k(x) \geq 0$ for $x \in K$.
(ii) $f_k(x) \to 0$ pointwise in K.
(i) $f_k(x) \leq f_\ell(x)$ for $x \in K$ whenever $k \geq \ell$.
Prove that $f_k(x) \to 0$ uniformly on K.

Problem 3 Let (X, ρ) be a non-empty complete metric space. Let $f : X \to X$ be such that there exists $\lambda \in (0, 1)$ satisfying

$$\rho(f(x), f(y)) \leq \lambda \rho(x, y) \quad \text{for all} \quad x, y \in X.$$

Prove that there exists a unique point $u \in X$ such that $f(u) = u$. (Hint: let $x \in X$, and consider the sequence $x,\ f(x),\ f(f(x)), ...$). This result is known as Banach's fixed-point theorem. It implies several existence theorems in the theory of partial differential equations.

Problems from Royden:

Chapter 7, Section 8: problem 43.
Chapter 7, Section 10: problems 47, 50.

(∗) **Problem.** Let (X, ρ) be a metric space, and let

$$C_1(X) = \{f : X \to \mathbb{R},\ f \text{ continuous} : \sup_{x \in X} |f(x) - \rho(x, x_0)| < \infty\},$$

for some chosen $x_0 \in X$. For $f, g \in C_1(X)$, define

$$\sigma(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

1. Show that $(C_1(X), \sigma)$ is a complete metric space which is independent of the choice of x_0.
2. Define the map $\iota : X \to C_1(X)$ by $\iota(x) = \rho_x$, where ρ_x denotes the function $\rho_x(y) = \rho(x, y)$. Show that ι is an isometric embedding of X into $C_1(X)$.