Math 524

Homework due 02/28/01

Problem 1.(Gagliardo-Nirenberg-Sobolev inequality) Let $1 \leq p < n$, and let $B \subset \mathbb{R}^n$ denote the ball of center 0 and radius 1. Show that there exists a constant C depending only on n and p such that

$$\|u\|_{\frac{pn}{n-p}} \leq C\|Du\|_p,$$

for all $u \in C_c^1(B)$. Here

$$\|f\|_q = \left(\int_B |f|^q\right)^{\frac{1}{q}}.$$

Hint: Recall the fundamental theorem of calculus and consider the case $p = 1$ first.

Problem 2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $p \geq 2$.

2.1. Show that for $f, g \in L^p(\mu)$,

$$\|\frac{f+g}{2}\|_p^p + \|\frac{f-g}{2}\|_p^p \leq \frac{1}{2}\|f\|_p^p + \frac{1}{2}\|g\|_p^p.$$

2.2. Let $\{f_n\} \subset L^p(\mu)$ and $f \in L^p(\mu)$. Suppose that $\|f_n\|_p = \|f\|_p = 1$ and that

$$\lim_{n\to\infty} \|\frac{f_n + f}{2}\|_p = 1.$$

Show that

$$\lim_{n\to\infty} \|f_n - f\|_p = 0,$$

i.e. $L^p(\mu)$ is *locally uniformly convex*.

Problem 3. (Radon-Riesz Theorem) Let $\{f_n\} \subset L^p(\mu)$ and $f \in L^p(\mu)$. Let $p \geq 2$. Suppose that $f_n \rightharpoonup f$ in $L^p(\mu)$ and that $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$. Prove that

$$\lim_{n\to\infty} \|f_n - f\|_p = 0.$$

Hint: Use problem 2.

Problem 4. Let $n \geq 3$. Let $B \subset \mathbb{R}^n$ denote the ball of center 0 and radius 1. Let $P = \varphi X$ where $\varphi \in C^1(B)$ and $X \in C^1(B, \mathbb{R}^n)$.

4.1. Show that

$$\operatorname{div} \mathrm{P} = \varphi \operatorname{div} \mathrm{X} + \langle \nabla \varphi, \mathrm{X} \rangle.$$

4.2. Let u be a harmonic function in B, i.e. $\Delta u = 0$ in B. Denote by B_r the ball of center 0 and radius r. Show that for $r \in (0, 1)$

$$(*) \qquad r \int_{\partial B_r} |\nabla u|^2 = 2r \int_{\partial B_r} \langle \nabla u, \nu \rangle^2 + (n-2) \int_{B_r} |\nabla u|^2,$$

where $\nu = \frac{X}{|X|}$, and X denotes the position vector.
Hint: Consider

$$\operatorname{div} (\mathrm{X} |\nabla \mathrm{u}|^2) - 2 \operatorname{div} (\langle \mathrm{X}, \nabla \mathrm{u} \rangle \nabla \mathrm{u}).$$

Remark: The monotonicity formula above $(*)$ is an useful tool when studying the order of vanishing of harmonic functions.