Math 524

Homework due 01/24/01

Problem 1. Let $f : [-1, 1] \to \mathbf{R}$ be a Lipschitz function; that is, there exists $M > 0$ so that

$$|f(x) - f(y)| \leq M|x - y| \quad \forall\ x,\ y \in [-1, 1].$$

1.1 Show that $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ exists a.e. $x \in [-1, 1]$.

1.2 Let $Z = \{x \in [-1, 1] :\ f(x) = 0\}$. Prove that $f'(x) = 0$ a.e. $x \in Z$.

Problem 2. Show that there exists a constant $C > 0$ such that for every $f \in L^1_{loc}(\mathbb{R}^n)$

$$m\left(\{x \in \mathbb{R}^n : Mf(x) > \alpha\}\right) \leq \frac{C}{\alpha} \int_{\{x:\ |f(x)| \geq \frac{\alpha}{2}\}} |f(y)|\, dy.$$

Recall that for $x \in \mathbb{R}^n$

$$Mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\, d(y).$$

Problem 3. Let $p \geq 1$, and let $f : \mathbb{R}^n \to [0, \infty)$ be measurable. Let

$$A_t = \{x \in \mathbb{R}^n : f(x) > t\}.$$

Show that

$$\int_{A_0 \times (0,\infty)} t^{p-1} \chi_{A_t}(x)\, dx\, dt = \int_0^\infty t^{p-1} m(A_t)\, dt = \frac{1}{p} \int_{A_0} f^p(x)\, dx$$

Problem 4. For $p > 1$, define

$$L^p(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R} : \int_{\mathbb{R}^n} |f(y)|^p \, dy < \infty \right\}.$$

Show that there exists a constant $C(p) > 1$ depending on p such that

$$\left(\int_{\mathbb{R}^n} [Mf(x)]^p \, dx \right)^{\frac{1}{p}} \leq C(p) \left(\int_{\mathbb{R}^n} |f|(x)^p \, dx \right)^{\frac{1}{p}},$$

for all $f \in L^p(\mathbb{R}^n)$. (Hint: Use problems 2 and 3.)

Problem 5. For $f \in L^1_{loc}(\mathbb{R}^n)$, and $0 < s < n$ define

$$\Lambda_s = \{ x \in \mathbb{R}^n : \liminf_{r \to 0} \frac{1}{r^s} \int_{B(x,r)} |f|(y) \, dy > 0. \}$$

Prove that $m(\Lambda_s) = 0$, for all $s \in (0, n)$.

Problems from Folland

Chapter 3, Section 4 : problems 23, 25.