Construction of rational and real powers

The goal this problem is to construct the rational and the real powers of any positive real number. We start by recalling what we know about the integer powers of a real number. By definition if $x \in \mathbf{R}$, and $x \neq 0$, $x^0 = 1$, and for $n \in \mathbf{N}$, $x^n = xx^{n-1}$. Note that in particular the associativity of the multiplication in $\mathbf{R}$ implies that

$$x^{n+m} = x^n x^m, \quad \text{and} \quad (x^n)^m = x^{nm} \quad \text{for} \quad n, m \in \mathbf{N}. \tag{1}$$

We denote $1/x^n$ by x^{-n}. By uniqueness of the inverse element, and the fact that the commutativity of the multiplication in $\mathbf{R}$ implies that $(1/x)^n x^n = (x^{-1})^n x^n = 1$ we conclude that

$$(x^{-1})^n = 1/x^n = x^{-n}. \tag{2}$$

Combining the first part of (1) and equality (2) it is easy to show that

$$x^{n+m} = x^n x^m \quad \text{for} \quad n, m \in \mathbf{Z}. \tag{3}$$

Fix $b > 1$.

(a) If $m, n, p, q \in \mathbf{Z}$, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}. \tag{4}$$

Hence it makes sense to define $b^r = (b^m)^{\frac{1}{n}}$

Proof: Since $b > 1$, $b^p > 1$ and $b^m > 1$. $(b^m)^{\frac{1}{n}}$ denotes the unique positive n^{th} root of b^m, and $(b^p)^{\frac{1}{q}}$ denotes the unique positive q^{th} root of b^p. Note that the second equality in (1) implies that

$$\left[(b^m)^{\frac{1}{n}}\right]^{nq} = \left(\left[(b^m)^{\frac{1}{n}}\right]^n\right)^q = (b^m)^q = b^{mq},$$

and

$$\left[(b^p)^{\frac{1}{q}}\right]^{nq} = b^{np}.$$

Since $r = m/n = p/q$ then $mq = np \in \mathbf{Z}$ and $b^{mq} = b^{np}$. Therefore $(b^m)^{\frac{1}{n}}$, and $(b^p)^{\frac{1}{q}}$ are positive $(nq)^{th}$ roots of b^{np}. Since the positive $(nq)^{th}$ root of b^{np} is unique we conclude that

$$(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}. \qquad \blacksquare$$

(b) Prove that if r and s are rational, then

$$b^{r+s} = b^r b^s. \tag{5}$$

Proof: Let $r = m/n$ and $s = p/q$, where $m, n, p, q \in \mathbf{Z}$, $n > 0$, $q > 0$. Thus $r + s = (mq + np)/nq$, and

$$\begin{aligned}
b^{r+s} &= (b^{mq+np})^{\frac{1}{nq}} && \text{by the definition in } (a)\\
&= (b^{mq}b^{np})^{\frac{1}{nq}} && \text{by (3)}\\
&= (b^{mq})^{\frac{1}{nq}}(b^{np})^{\frac{1}{nq}} && \text{by the uniqueness of the } nq \text{ root}\\
&= (b^m)^{\frac{1}{n}}(b^p)^{\frac{1}{q}} && \text{by } (a)\\
&= b^r b^s && \text{by the definition in } (a).
\end{aligned}$$

(c) If $x \in \mathbf{R}$, let

$$B(x) = \{b^t : t \in \mathbf{Q} \text{ and } t \le x\}.$$

Prove that if r is rational

$$b^r = \sup B(r). \tag{6}$$

Hence it makes sense to define for every $x \in \mathbf{R}$,

$$b^x = \sup B(x). \tag{7}$$

Proof: First note that since $b > 1$, if $n \in \mathbf{N}$ then $b^n > 1$ (see proof of (a)), and $b^{1/n} > 1$. If $b^{\frac{1}{n}} \le 1$, then $(b^{\frac{1}{n}})^n \le 1$, i.e. $b \le 1$, which contradicts the initial assumption that $b > 1$. In order to prove (6) note that if r is rational $b^r \in B(r)$ thus $B(r) \neq \emptyset$. Let $t \le r$ and assume that $r = m/n$ and $t = p/q$ with $m, n, p, q \in \mathbf{Z}$, $n > 0$, $q > 0$. Then $r - t = (mq - np)/nq \ge 0$, and $b^{r-t} = (b^{mq-np})^{\frac{1}{nq}}$. Since $mq - np \ge 0$, $b^{mq-np} \ge 1$, and $b^{r-t} \ge 1$ (see remark above). Thus, by (5), $b^r b^{-t} \ge 1$. But $b^t b^{-t} = (b^p)^{1/q}(b^{-p})^{1/q} = (b^p b^{-p})^{1/q} = 1$. The uniqueness of the inverse guarantees then that $b^{-t} = 1/b^t > 0$. This combine with the fact that $b^r b^{-t} \ge 1$ and Proposition 1.18(b) implies that

$$b^t \le b^r \quad t,\ r \in \mathbf{Q} \quad \text{whenever} \quad \text{and} \quad t \le r. \tag{8}$$

In particular this implies that $B(r) \neq \emptyset$ is bounded above. Thus the least-upper-bounded property of $\mathbf{R}$ insures that $\sup B(r)$ exists. Hence since $b^r \in B(r)$, $b^r \le \sup B(r)$. (8) shows that b^r is an upper bound for $B(r)$ thus $b^r = \sup B(r)$ ■

In order to define $b^x = \sup B(x)$ for any $x \in \mathbf{R}$, we should show that $B(x)$ is not empty and bounded above. The Archimedean property guarantees that exist $n, m \in \mathbf{N}$ such that $-m < x < n$. Thus $b^{-m} \in B(x)$, i.e. $B(x) \neq \emptyset$. If t rational and $t \leq x < n$ (8) implies that $b^t \leq b^n$, i.e. $B(x) \neq \emptyset$ is bounded above. Thus by the least-upper-bounded property of $\mathbf{R}$, $\sup B(x)$ exists, and we define $b^x = \sup B(x)$. Note that $b^x > 0$.

(d) Prove that for all x, $y \in \mathbf{R}$,

$$b^{x+y} = b^x b^y. \tag{9}$$

Proof: In order to show (9) we first prove that $b^{x+y} \geq b^x b^y$. Let $r,\ s \in \mathbf{Q}$, $r \leq x$, $s \leq y$, then $r+s \in \mathbf{Q}$, $r+s \leq x+y$ and by (5) and (7) $b^r = b^{r+s}b^{-s} \leq b^{x+y}b^{-s}$. Hence for any $s \in \mathbf{Q}$, $s \leq y$, $b^{x+y}b^{-s}$ is an upper bound for $B(x)$. Since $b^x = \sup B(x)$, $b^x \leq b^{x+y}b^{-s}$ for any $s \in \mathbf{Q}$, with $s \leq y$. Thus for any $s \in \mathbf{Q}$, with $s \leq y$, $b^s \leq b^{x+y}(1/b^x)$, i.e $b^{x+y}(1/b^x)$ is an upper bound for $B(y)$. Therefore $b^y \leq b^{x+y}(1/b^x)$, which implies that $b^x b^y \leq b^{x+y}$. Assume $b^x b^y < b^{x+y}$, then $b^x b^y$ is not an upper bound for $B(x+y)$, i.e. there exists $t \in \mathbf{Q}$, $t \leq x+y$, and $b^x b^y < b^t$. We may assume that $t < x+y$ (see justification below). Since $\mathbf{Q}$ is dense in $\mathbf{R}$, and $t - x < y$ there exists $s \in \mathbf{Q}$ such that $t - x < s < y$. Let $r = t - s \in \mathbf{Q}$, note that $r = t - s < (x+s) - s = x$ and $r + s = t$. Thus the assumption above, (5) and (7) imply that

$$b^x b^y < b^t = b^{r+s} = b^r b^s \leq b^x b^y,$$

which is a contradiction. Therefore $b^{x+y} = b^x b^y$.

We have finished the proof of (d) provided that we justify the fact that $t < x+y$ in the second part of the argument above. In order to do this we follow the first few steps in problem 7. For $b > 1$, and $n \in \mathbf{N}$ $b^n - 1 = (b-1)(b^{n-1} + ... + 1) > n(b-1)$. Since $b > 1$, then $b^{1/n} > 1$ as shown in the proof of (c). Applying the previous inequality to $b^{1/n}$ in place of b we conclude that $b - 1 > n(b^{1/n} - 1)$. A simple computation shows that if $\tau > 1$ and $n > (b-1)/(\tau - 1)$ then $b^{1/n} < \tau$. Now recall the argument above : *there exists* $t \in \mathbf{Q}$, $t \leq x+y$, *and* $b^x b^y < b^t$. Since $\mathbf{Q}$ is dense in $\mathbf{R}$ there exists $p \in \mathbf{Q}$ such that $b^x b^y < p < b^t$. Let $\tau = b^t/p > 1$. The remark above guarantees that there exists $n \in \mathbf{N}$ such that $b^{1/n} < \tau = b^t/p$. Thus $b^x b^y < p < b^{t-1/n}$ and $t - 1/n < x + y$. ■