On Conformal Welding and Quasicircles

S. ROHDE

1. Introduction

Let C be a quasicircle (i.e., the image of a circle under a quasiconformal mapping) and let G_0, G_∞ be the bounded and unbounded components of $\hat{\mathbf{C}} \setminus C$. Throughout this paper we will assume that $0 \in G_0$. By ω_0, ω_∞ we denote the harmonic measures on C, evaluated at $0, \infty$. We consider the conformal mappings $f: \mathbf{D} \to G_0$, $f(0)=0$ and $g: \mathbf{D} \to G_\infty$, $g(0)=\infty$, where $\mathbf{D}$ is the unit disc $\{|z|<1\}$. The welding $\varphi: \mathbf{T} \to \mathbf{T}$ is defined by

$$\varphi(\zeta) = (g^{-1} \circ f)(\zeta), \quad \zeta \in \mathbf{T}, \tag{1}$$

where $\mathbf{T}$ is the unit circle $\{|z|=1\}$. Since C is a quasicircle, the welding φ is quasisymmetric.

We are interested in quasicircles C that are "far away from being smooth." For $w_1, w_2 \in C$ let $\langle w_1, w_2 \rangle$ denote the smaller subarc of C with endpoints w_1, w_2. We define

$$\beta(C) = \inf_{w_1, w_2 \in C} \sup_{w \in \langle w_1, w_2 \rangle} \frac{|w_1 - w| + |w_2 - w|}{|w_1 - w_2|}. \tag{2}$$

Clearly $\beta(C) \geq 1$, and since C is a quasicircle the right-hand side of (2) remains bounded if we replace inf by sup. If C has a tangent at some point $w \in C$, then $\beta(C) = 1$. Of course there are quasicircles C with $\beta(C) > 1$, for example the snowflake. Other examples are given in Section 3.

We will use the abbreviation dim for Hausdorff dimension.

THEOREM. *Let C be a quasicircle with $\beta(C) > 1$. Then there is a set $E \subset \mathbf{T}$ with*

$$\dim E < 1 \quad \textit{and} \quad \dim \varphi(\mathbf{T} \setminus E) < 1. \tag{3}$$

Tukia [11] recently constructed quasisymmetric mappings φ satisfying (3). With the theorem we get a new class of examples.

The proof of the theorem relies on the following proposition.

PROPOSITION. *For any quasicircle C there are positive constants c, ϵ_0 and a number $\delta \geq 0$, where δ depends only on $\beta(C)$, such that the following*

Received January 18, 1990. Revision received August 13, 1990.
Michigan Math. J. 38 (1991).

is true: If D is a disc with $D\cap C\neq\emptyset$ and $\operatorname{diam} D\leq\epsilon_0$, *then*

$$\omega_0(D)\omega_\infty(D)\leq c(\operatorname{diam} D)^{2+\delta}. \tag{4}$$

Furthermore, we have

$$\delta>0 \quad \textit{for } \beta(C)>1. \tag{5}$$

The proposition is a generalization of the inequality

$$\omega_0(D)\omega_\infty(D)\leq c(\operatorname{diam} D)^2,$$

established in [2], where this inequality was used to show that φ is singular if C has no tangents. Hence we get a stronger form of singularity of φ if we assume that C is far away from having tangents, where this "distance" is measured by $\beta(C)$. Our proofs are similar to the proofs given in [2].

A result of Bishop shows that the condition $\beta(C)>1$ cannot be replaced by the condition $\dim C>1$. An example of a quasicircle C with $\dim C>1$ and a Lipschitz-continuous welding φ is given in [1].

I want to thank Ch. Pommerenke for our discussions and J. L. Fernández for directing my attention to Tukia's paper [11].

REMARK. The referee pointed out that the results of this paper have been known (unpublished) to some people working in this area. He also indicated a more elementary approach to the theorem, which avoids the application of Markarov's result; the proof is outlined in Section 3.

2. Proofs

Proof of Proposition. Let D be a disc of small radius ϵ with $D\cap C\neq\emptyset$. By D_n we denote the disc concentric with D and with radius $2^n\epsilon$. Let A_n be the annulus $D_{n+1}\backslash D_n$. We will use the notion of the extremal length of a curve family and Pflugers theorem (see, e.g., [9]). For this purpose consider two fixed curves $K_0\subset G_0$ and $K_\infty\subset G_\infty$, both enclosing the origin. Let Γ_0 (resp. Γ_∞) be the family of curves joining ∂D to K_0 (K_∞) in G_0 (G_∞). By Pflugers theorem we have

$$\omega_0(D)\omega_\infty(D)\leq c_1\exp[-\pi(\lambda(\Gamma_0)+\lambda(\Gamma_\infty))],$$

hence we are done if we prove

$$\lambda(\Gamma_0)+\lambda(\Gamma_\infty)\geq\frac{2+\delta}{\pi}\log\frac{1}{\epsilon} \tag{6}$$

for some constant δ.

To prove (6), let $\Gamma_0^{(n)}$ (resp. $\Gamma_\infty^{(n)}$) be the family of curves joining ∂D_n to ∂D_{n+1} in $A_n\cap G_0$ ($A_n\cap G_\infty$), $n=1,2,\ldots,N$, where N is the largest integer with the property that $A_n\cap K_0=\emptyset$ and $A_n\cap K_\infty=\emptyset$ for $n\leq N$. Hence $2^N\epsilon$ is comparable to $\operatorname{dist}(K_0\cup K_\infty, C)$, and this means

$$\frac{1}{\epsilon}\leq c_2 2^N. \tag{7}$$

Furthermore,

$$\lambda(\Gamma_0)+\lambda(\Gamma_\infty)\geq\sum_{n=1}^{N}[\lambda(\Gamma_0^{(n)})+\lambda(\Gamma_\infty^{(n)})]. \tag{8}$$

Hence (6) follows from (7) and (8) if we prove

$$\lambda(\Gamma_0^{(n)})+\lambda(\Gamma_\infty^{(n)})\geq\frac{2+\delta}{\pi}\log 2 \quad \text{for each } n=1,2,\dots,N. \tag{9}$$

In order to prove this, fix n and consider the curve family $\Gamma=\Gamma_0^{(n)}\cup\Gamma_\infty^{(n)}$. Then

$$\lambda(\Gamma)^{-1}=\lambda(\Gamma_0^{(n)})^{-1}+\lambda(\Gamma_\infty^{(n)})^{-1};$$

hence

$$\lambda(\Gamma_0^{(n)})+\lambda(\Gamma_\infty^{(n)})\geq 4\lambda(\Gamma). \tag{10}$$

Since C meets D it meets every A_n, and therefore there exists a subarc C' of C lying in A_n and joining the boundary components of A_n. Let A_n' denote the domain $A_n\backslash C'$ and let Γ' be the family of curves joining ∂D_n with ∂D_{n+1} in A_n'. Clearly $\Gamma'\supset\Gamma$ so that $\lambda(\Gamma)\geq\lambda(\Gamma')$. Since $\beta(C)>1$, the domain A_n' does not contain a sector

$$\{2^n\epsilon<|z|<2^{n+1}\epsilon,\ \alpha<\arg(z-z_0)<\beta\}$$

with $\beta-\alpha$ arbitrarily close to 2π (here z_0 is the center of D).

We apply a result of Carleson [5] to obtain

$$\lambda(\Gamma')\geq\frac{1+\delta/2}{2\pi}\log 2$$

for some fixed number $\delta>0$, depending only on $\beta(C)$. Together with (10) this gives the desired inequality (9). □

Proof of Theorem. Let δ be the number given by the proposition. By (5) we have $\delta>0$. We will prove (3) with the set

$$E=\{\zeta\in\mathbf{T}\mid\limsup_{r\to 1}|f'(r\zeta)|(1-r)^{\delta/(2+\delta)}>0\}.$$

By a result of Makarov [7],

$$\dim E\leq\alpha(\delta)<1,$$

where $\alpha(\delta)$ is independent of f.

Let $\zeta\in\mathbf{T}\backslash E$ be given. Then $|f'(r\zeta)|<(1-r)^{-\delta/(2+\delta)}$ for $r\geq r_0(\zeta)$. Since C is a quasicircle, there is a constant c_3, depending only on f, such that

$$c_3(1-r)|f'(r\zeta)|\leq\operatorname{diam}f(I_r(\zeta))\leq\frac{1}{c_3}(1-r)|f'(r\zeta)|, \tag{11}$$

where $I_r(\zeta)$ is the arc on $\mathbf{T}$ with midpoint ζ and length $2\pi(1-r)$ (see [7]).

With $\epsilon=\operatorname{diam}f(I_r(\zeta))$, from (11) and $(1-r)|f'(r\zeta)|<(1-r)^{2/(2+\delta)}$ we obtain that

$$\omega_0(f(I_r(\zeta)))=1-r\geq c_4\epsilon^{1+\delta/2}\quad\text{for } r\geq r_0(\zeta).$$

Together with (4) this gives

$$\omega_\infty(f(I_r(\zeta))) \le \frac{c}{c_4}\epsilon^{1+\delta/2}. \tag{12}$$

Let $\xi = \varphi(\zeta)$ and $\rho = 1 - \omega_\infty(f(I_r(\zeta)))$. There is a point $\xi' \in \mathbf{T}$ with $I_\rho(\xi') = \varphi(I_r(\zeta))$. By the Koebe distortion theorem we have

$$|g'(\rho\xi)| \ge c_5 |g'(\rho\xi')|. \tag{13}$$

We obtain from (11) with g, ρ, ξ' instead of f, r, ζ and from (12) that

$$(1-\rho)|g'(\rho\xi')| \ge c_3\epsilon \ge c_3\left[\frac{c_4}{c}(1-\rho)\right]^{2/(2+\delta)}.$$

Together with (13) we have

$$|g'(\rho\xi)| \ge c_6(1-\rho)^{-\delta/(2+\delta)} \quad \text{for } \rho > \rho_0(\xi).$$

Hence

$$\varphi(\mathbf{T}\setminus E) \subset \{\zeta \in \mathbf{T} \mid \limsup_{\rho\to 1} |g'(\rho\xi)|(1-\rho)^{\delta/(2+\delta)} > 0\},$$

and again (see [7]) the set on the right-hand side is of dimension $\le \alpha(\delta) < 1$. □

3. Examples and Remarks

(a) Let C be the Julia set of the polynomial $z^2 + \lambda z$, $0 < |\lambda| < 1$. It is known that C is a quasicircle. It has been shown in [10] that the conformal mapping $f: \mathbf{D} \to G_0$ has the worst possible behaviour in the following sense:

$$\limsup_{r\to 1} \frac{\log|f'(r\zeta)|}{\sqrt{\log[1/(1-r)] \log\log\log[1/(1-r)]}} > 0 \quad \text{for a.e. } \zeta \in \mathbf{T} \tag{14}$$

(see, e.g., [8] for a detailed discussion of this and its consequences).

Jones [6] has shown that (14) also holds for quasicircles C with $\beta(C) > 1$, hence without using any dynamical structure on C. The starting point of our paper was the observation that it is easy to prove a weaker form of (3) for Julia sets by using ergodic theory. The argument is as follows: Since f maps 0 to the attractive fixed point of p_λ (which is again 0), the composition $f^{-1} \circ p_\lambda \circ f$ is a Blaschke product of degree 2, fixing the origin. Similarly, $(g^{-1} \circ p_\lambda \circ g)(z) = z^2$. It follows from [10] that

$$h(p_\lambda, \omega_0) < \log 2 = h(p_\lambda, \omega_\infty),$$

where $h(p_\lambda, \mu)$ is the entropy of p_λ with respect to the invariant probability measure μ on C.

Now consider the repelling fixed point $\zeta_0 \in C$. The set $p_\lambda^{-n}(\zeta_0)$ consists of 2^n points of C, dividing C into 2^n arcs $I_k^{(n)}$. For $\zeta \in C$ let us denote by $I_n(\zeta)$ the arc $I_k^{(n)}$ containing ζ. Then by the Shannon–McMillan–Brieman theorem we have

$$\frac{1}{n}\log \omega_0(I_n(\zeta)) \to -h(p_\lambda, \omega_0), \quad n\to\infty, \ \omega_0\text{-a.e. on } C,$$

hence $\omega_0(I_n(\zeta)) \geq a^n$ for some fixed number $a > \frac{1}{2}$ for ω_0-a.e. $\zeta \in C$ and $n \geq n_0(\zeta)$. On the other hand, it is easy to see that $\omega_\infty(I_n(\zeta)) = 2^{-n}$. From this it is not difficult to prove that there is a set $E \subset C$ with $\omega_0(E) = 1$ but $\dim g^{-1}(E) < 1$.

(b) Let Γ_1, Γ_2 be finitely generated Fuchsian groups of the first kind and let $\alpha: \Gamma_1 \to \Gamma_2$ be an isomorphism. Let $\varphi: \mathbf{T} \to \mathbf{T}$ be a homeomorphism satisfying $\varphi\circ\gamma\circ\varphi^{-1} = \alpha\gamma$ ($\gamma \in \Gamma_1$). Tukia asked in [11] whether (3) is valid for φ if φ is not a Möbius transformation (in that case it is known that φ is singular). We will give a partial answer to this problem. In full generality this problem has been solved (independently) in [3].

Let us assume that Γ_1 has a compact fundamental domain F and let Γ be the quasi-Fuchsian group associated to $(\Gamma_1, \Gamma_2, \alpha)$. Let C be the quasicircle invariant under Γ and let f, g, G_0, G_∞ be as above. Then $\varphi = g^{-1}\circ f$ satisfies $\varphi\circ\gamma\circ\varphi^{-1} = \alpha(\gamma)$ ($\gamma \in \Gamma_1$), hence (3) is valid if $\beta(C) > 1$. The Hausdorff dimension of C is studied in [4].

Jones [6] has shown, using Caratheodory kernel convergence, that $\beta(C) > 1$ holds if the following condition is satisfied: There are positive constants c and ρ such that in each hyperbolic disc (in $\mathbf{D}$) of radius ρ there is a point z such that $|S_f(z)|(1-|z|^2)^2 \geq c$. By assumption, φ is not absolutely continuous and hence C is not rectifiable; it follows that $|S_f(z_0)|(1-|z_0|^2)^2 > 0$ at some point $z_0 \in F$. Let ρ be the hyperbolic diameter of F. If $z \in \mathbf{D}$ then $z = \gamma(z')$ with $z' \in F$ and $\gamma \in \Gamma_1$, and $\gamma(z_0)$ lies in the hyperbolic disc of radius ρ around z. Furthermore, $|S_f(\gamma(z_0))|(1-|\gamma(z_0)|^2)^2 = |S_f(z_0)|(1-|z_0|^2)^2$; hence the condition of Jones is satisfied.

(c) We now outline the referee's alternate proof of the theorem.

LEMMA (referee). *For any quasicircle C with $\beta(C) > 1$ there is an integer N and a $\epsilon > 0$ such that, for any arc $I \subset \mathbf{T}$ and the N disjoint subarcs I_j of equal length, the inequality*

$$\sum_{j=1}^{N} |I_j|^{1/2}|\varphi(I_j)|^{1/2} \leq (1-2\epsilon)|I|^{1/2}|\varphi(I)|^{1/2}$$

holds.

This lemma can be proven by contradiction. If the lemma were not true then there would be a sequence ϵ_N tending to zero, and a corresponding sequence of arcs such that equality holds in the lemma. After rescaling these arcs—with the aid of some Möbius transformation—to length (say) 1/2, the compactness of quasisymmetric maps leads to a quasisymmetric limit function ψ with associated quasicircle Γ', such that the inequality of the lemma is an equality for some arc I and all integers N. Hence the mapping $t \mapsto \arg\psi(e^{it})$ would be linear on some interval and it would follow that the corresponding subarc of Γ' is smooth, in contradiction to $\beta(\Gamma') \geq \beta(\Gamma)$. The last inequality is again easily proven using compactness of quasisymmetric maps.

It follows from the lemma that there is a number $\delta > 0$ such that, with $\mu = (1-\delta)/2$,

$$\sum_{j=1}^{N} |I_j|^\mu |\varphi(I_j)|^\mu \leq (1-\epsilon)|I|^\mu |\varphi(I)|^\mu.$$

It is not difficult to show that any function φ satisfying the last inequality for some fixed numbers N, ϵ, δ and any arc I has the property discussed in our Theorem (see [3] for the details.)

References

1. C. Bishop, *A counterexample in conformal welding concerning Hausdorff dimension,* Michigan Math. J. 35 (1988), 151–159.
2. C. Bishop, L. Carleson, J. Garnett, and P. Jones, *Harmonic measures supported on curves,* Pacific J. Math. 138 (1989), 233–236.
3. C. Bishop and T. Steger, *Representation theoretic rigidity in* PSL(2, **R**), preprint.
4. R. Bowen, *Hausdorff dimension of quasicircles,* Inst. Hautes Études Sci. Publ. Math. 50 (1979), 11–25.
5. L. Carleson, *On the distortion of sets on a Jordan curve under conformal mapping,* Duke Math. J. 40 (1973), 547–559.
6. P. Jones, *Square functions, Cauchy integrals, analytic capacity, and harmonic measure,* Lecture Notes in Math., 1384, pp. 24–68, Springer, Berlin, 1989.
7. N. G. Makarov, *Conformal mapping and Hausdorff measures,* Ark. Mat. 25 (1987), 41–89.
8. ———, *Probability methods in the theory of conformal mappings,* (Russian), Algebra i Analiz 1 (1989), 3–59.
9. M. Ohtsuka, *Dirichlet problem, extremal length and prime ends,* Van Nostrand Reinhold, New York, 1970.
10. F. Przytycki, M. Urbański, and A. Zdunik, *Harmonic, Gibbs and Hausdorff measures on repellers for holomorphic maps, I,* Ann. of Math. (2) 130 (1989), 1–40.
11. P. Tukia, *Hausdorff dimension and quasisymmetric mappings,* Math. Scand. 65 (1989), 152–160.

Tech. Univ. of Berlin
Sekr. MA8-2
Strasse d. 17. Juni 136
1000 Berlin 12
Germany