Some remarks on Laplacian growth

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1 Introduction

In this paper, we will discuss the Hastings-Levitov model $HL(\alpha)$ for random growing clusters K_n, where $0 \le \alpha \le 2$ is a parameter, as well as modifications of this model. Background on this model and its physical meaning is provided in Section 2. The main object of this paper is a regularization $RHL(\alpha)$ defined as follows (see Section 3 for the motivation and interpretation and Section 4 for a picture). Let K_1 be the unit disc and construct K_{n+1} from K_n by attaching a new "particle" via conformal maps: If ϕ_n denotes the conformal map from $\Delta = \{|z| > 1\}$ onto $\mathbb{C} \setminus K_n$, then $\phi_{n+1} = \phi_n \circ h_n$ with a random conformal map h_n from Δ into Δ. The natural choice is $h_n(z) = u_n h_{\delta_n}(z/u_n)$ with a randomly chosen point u_n on the unit circle and the conformal map h_d of $\Delta \setminus [1, 1+d]$. The "random size" δ_n of the new particle is defined by first setting

$$\epsilon_t(u, \delta) = \inf\{\epsilon > 0;\ \epsilon |\varphi_t'((1+\epsilon)u)| = \delta\},$$

($\delta > 0$ is small and kept fixed) and then setting

$$\delta_n = \delta^{1-\alpha/2} \epsilon_n(u_n, \delta)^{\alpha/2}.$$

Thus for $\alpha = 0$ we compose random rotations of a fixed map, whereas for $\alpha = 2$ we have a version of diffusion limited aggregation DLA.

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We prove the existence of the scaling limit of $HL(0)$ (Theorem 1) and describe this limit in terms of the Löwner equation in Section 4. We then analyze the size of the $RHL(\alpha)$-cluster. In Section 5 we show that for $\alpha = 0$ the Hausdorff dimension is 1 (Theorems 2 and 3). In Section 6 we show that, with the appropriate interpretation, the dimension behaves at most like $1+2\alpha$ for small α (Theorem 5). In Section 7 we prove that the "dimension" near $\alpha = 2$ is strictly greater than 1. We finally consider the deterministic limiting version, following Carleson and Makarov, in Section 8.

2 The Hastings-Levitov model

In the last few decades, physicists have observed that many phenomena lead to objects of similar fractal geometry. Typically, these phenomena show growing clusters of various kinds. They appear for instance as electrodeposition, colloidal aggregation, in the study of lightnings or cracks, in tumoral growth or bacterial colonies .

Beginning with Eden in the 60's, the models proposed to explain these phenomena are random growth models. This means that we model the process as an increasing sequence of compact sets K_n in space, K_{n+1} being obtained from K_n by randomly choosing a point on the boundary ∂K_n and by attaching a given object at this point. The questions are of course

1) What is the probability law for the choice of the boundary point?

2) What object do we attach?

This problem makes sense in any dimension. However, we will exclusively deal with the planar case where we can use the powerful tools of geometric function theory.

Even if all the apparently unrelated experiments show the same rough shapes, some differences appear: clusters are nearly round in Eden's model while they look almost arcwise affine in the lightning situation. This means that some parameters are needed in the answers of the two questions in order to explain the differences. This is precisely the purpose of the Dielectric breakdown model (DBM), a one parameter family of growth processes defined as follows:

1) The probability law for the choice of the boundary point is given by

$$dP_\eta = \frac{|\nabla G_n(x)|^\eta |dx|}{\int_{\partial K_n} |\nabla G_n(x)|^\eta |dx|}$$

where G_n is Green's function (potential) of K_n with pole at ∞ and $|dx|$ is arclength.

2) The added object is a fixed one (a disk or a segment for instance).

The parameter η varies from 0 to 1.

The case $\eta = 0$ is Eden's model. The boundary point is chosen under uniform (wrt arclength) law and simulations show round clusters.

The opposite case $\eta = 1$ is a version of DLA (diffusion limited aggregation). The probability law is harmonic measure with pole at infinity which is also known to be hitting probability for Brownian motion: this makes this model a very natural one for all phenomena with diffusing particles such as electrodeposition for instance.

Models based on diffusing particles are called Laplacian. Equivalently, they are the models for which the probability law is harmonic measure at infinity (but the objects may change as we shall see, so that Laplacian growth models generalize DLA).

Laplacian growth models are also easy to simulate. In the DBM family, only the $\eta = 1$-case gives rise to such a model. This is the reason why Hastings and Levitov [1] proposed a one-parameter family of Laplacian growth models as a substitute to DBM family.

We are now describing the Hastings-Levitov model HL(α). A general method to generate an increasing sequence of (random) connected sets K_n is as follows: Assuming that $\mathbb{C}\backslash K_n$ is connected, there exists a unique Riemann mapping (that is, holomorphic and bijective) $\varphi_n : \{|z| > 1\} \longrightarrow \mathbb{C}\backslash K_n$, normalized so that at infinity we have

$$\varphi_n(z) = c_n z + a_n + \frac{b_n}{z} + ..., \; c_n > 0. \tag{2.1}$$

The image of (normalized) Lebesgue measure of the unit circle under φ_n is precisely harmonic measure with pole at infinity, so that choosing a point on ∂K_n at random with respect to harmonic measure is the same as choosing $x = \varphi_n(u)$ where u is chosen at random on the unit circle with respect to arclength. Now K_{n+1} is constructed by defining φ_{n+1} directly as

$$\varphi_{n+1} = \varphi_n \circ h_n,$$

where $h_n(z) = u_n h_{\delta_n}(z/u_n)$, u_n being the randomly chosen point on the unit circle, δ_n a positive real parameter, and h_δ is a conformal map from $\{|z| > 1\}$ into itself. Obvious choices for h_δ are

– (strike model) the Riemann map between $\{|z| > 1\}$ and $\{|z| > 1\}\backslash[1, 1+\delta]$.

– (bump model) the Riemann mapping between $\{|z| > 1\}$ and $\{|z| > 1\}\backslash D_\delta$, D_δ being the closed disk centered on the positive real axis orthogonal to the unit disk and with radius δ.

This way of defining the growth has the advantage of being rigorous: in the DBM model it may be impossible to stick a given object at a given point or it can be done in several different manners. The disadvantage is that we do not control the shape of $K_{n+1}\backslash K_n$; only the size can be controled by an appropriate choice of the parameter δ_n.

Hastings and Levitov defined for $0 \leq \alpha \leq 2$ the Laplacian growth model (strike or bump) $HL(\alpha)$ by

$$\delta_n = \delta|\varphi_n'(u_n)|^{-\alpha/2}. \tag{2.2}$$

Heuristically, the size (diameter) of the "added object" is then of the order of $|\varphi_n'(u_n)|^{1-\alpha/2}$ (we think of δ as a small but fixed constant). This choice is made in order that local area growth expectation in the bump model is the same for DBM(η) and $HL(\eta+1)$, these authors believing that this assignment implies similar geometry. Thus $HL(\alpha)$ serves as a Laplacian substitute for DBM for the values $\alpha \in [1, 2]$, $\alpha = 2$ corresponding to DLA and $\alpha = 1$ to Eden's.

Even without physical significance, the models $HL(\alpha)$ for $\alpha \in [0,1]$ turn out to be interesting as we shall see. Moreover, HL observed an interesting phase transition at $\alpha = 1$, the growth passing at this point from a steady state to a turbulent one, at least as observed on simulations.

After modifying the HL model in the spirit of Carleson-Makarov [2], we will prove some rigourous results which confirm, if they do not prove, this observation. Before we proceed, we introduce deterministic growth models associated to random Laplacian models.

3 Deterministic models associated with HL growth models.

From now on we will always consider the "strike" model for $HL(\alpha)$. In this model it can easily be shown that

$$h_\delta(z) = (1+\lambda)z + ... \tag{3.1}$$

where

$$\lambda = \sqrt{1 + \frac{\delta^2}{4(1+\delta)}} - 1 \sim \frac{\delta^2}{8},\ \delta \text{ small.}$$

The deterministic model associated with this random model is obtained by first taking δ_0 small, obtaining the asymptotics (see [3])

$$\varphi_{n+1}(z) \sim \varphi_n(z) + \varphi_n'(z)\lambda_n z \frac{z+u_n}{z-u_n},$$

and then averaging over the unit circle. We then obtain the continuous equation

$$\frac{\partial}{\partial t}\varphi_t(z) = z\varphi_t'(z)\int_{\mathbb{T}} |\varphi_t'(u)|^{-\alpha}\frac{z+u}{z-u}\frac{|du|}{2\pi}.$$

This equation is closely related to the Löwner equation. Recall that if $(K_t)_{t\geq 0}$ is a (continuously) increasing sequence of full connected compact subsets of the plane, then there exists a one-parameter family (μ_t) of positive measures on the unit circle such that if φ_t denotes the Riemann mapping of $\mathbb{C}\backslash K_t$ then

$$\frac{\partial}{\partial t}\varphi_t(z) = z\varphi_t'(z)\int_{\mathbb{T}} \frac{z+u}{z-u} d\mu_t(u). \tag{3.2}$$

Conversely if (μ_t) is a one-parameter family of measures such that $t \mapsto \|\mu_t\|$ is absolutely continuous then there exists an increasing corresponding family (K_t) of compact subsets of the plane.

It follows that the $HL(\alpha)$ processes correspond to a Löwner process for which the family (μ_t) satisfies the equation

$$d\mu_t(u) = |\varphi_t'(u)|^{-\alpha}|du|. \tag{3.3}$$

Such an equation need not have a solution.

For $\alpha = 0$ it does. The solution is then simply

$$\varphi_t(z) = \varphi_0(e^t z)$$

and the growth process consists of the growth of K_0 by equipotentials. This is a very smoothing growth process.

For $\alpha = 2$ the equation (3.3) is the one driving Hele-Shaw flows and it is known to be ill-posed: in particular no solution exists if ∂K_0 is not real-analytic and even if a solution exists, it may stop to exist after some finite time.

As usual with ill-posed problems, some regularization is needed to ensure existence of solutions for all times. Recently Carleson and Makarov [2] proposed such a regularization, consisting in replacing $|\varphi_t'(u)|^{-2}$ by $\epsilon_t(u,\delta)^2$ in the equation (3.3), where

$$\epsilon_t(u,\delta) = \inf\{\epsilon > 0;\ \epsilon|\varphi_t'((1+\epsilon)u)| = \delta\}.$$

We recover the initial equation by taking formal limit as $\delta \to 0$ in the regularized equation with some time change.

We propose here to adopt this regularization for $HL(\alpha)$ growth. We will denote by $RHL(\alpha)$ the random growth model obtained by replacing the definition (2.2) of $HL(\alpha)$ by

$$\delta_n = \delta^{1-\alpha/2}\epsilon_n(u,\delta)^{\alpha/2}. \tag{3.4}$$

With this definition the models $RHL(0)$ and $RHL(2)$ really appear as "dual" models: For $RHL(0)$, at each step, we travel on a random external ray of the unit circle until we reach distance δ from the circle while for $RHL(2)$ we do the same thing but on a randomly chosen external ray of K_n until we reach distance δ from K_n (this last statement is a corollary of the Koebe distortion theorem).

We will also consider deterministic $RHL(\alpha)$ models; they are the Löwner processes such that

$$d\mu_t(u) = \epsilon_t(u,\delta)^{\alpha}|du|.$$

We conclude this paragraph by a remark; the Carleson-Makarov regularization is certainely not canonical and we will feel free to modify the definition of ϵ for our purposes. All the definitions that we will use will share the same feature: we recover $HL(\alpha)$ by taking formal limit as $\delta \longrightarrow 0$.

One of the questions of the theory is to find a "natural" regularization. In particular, the Carleson-Makarov regularization is very different from the usual "physical" regularizations consisting in adding surface tension.

4 The scaling limit of $HL(0)$

In this section, we will prove that the scaling limit of $HL(0)$ exists, and we will give two descriptions of this limit (one using the Löwner equation). For convenience, we will work with the strike model. Fix $\delta > 0$ and recall the definition of $HL(0)$: $K_n = \partial\phi_n(\Delta)$, where

$$\phi_n = h_1 \circ h_2 \circ \dots \circ h_n, \tag{4.1}$$

$h_j(z) = u_j h_\delta(z/u_j)$ with random $u_j \in \mathbb{T}$, and $h_\delta : \Delta \to \Delta \setminus [1,\delta]$ is the conformal map normalized by $h_\delta(z) = (1+\lambda)z + O(1)$ near ∞, for some

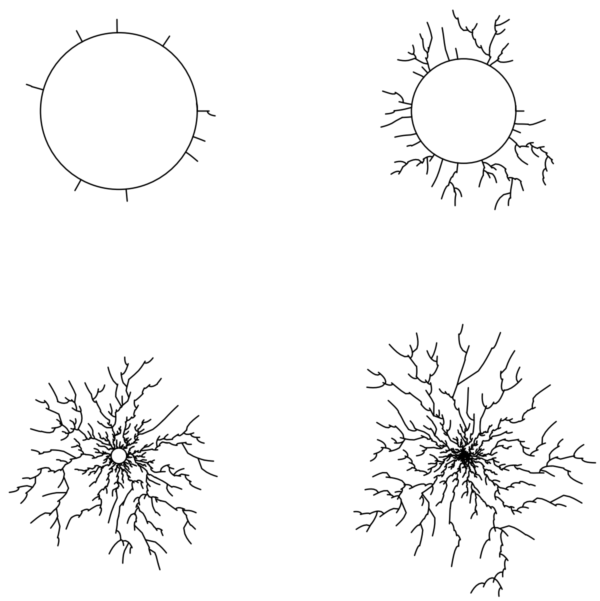

Figure 1: Samples of $\tilde{K}_n$ with $n = 10, 100, 500$ and 1000.

$\lambda > 0$. We will denote by $C_n = c(\phi_n) = (1+\lambda)^n$ the capacity of K_n, i.e. the z-coefficient of ϕ_n, and by $A_n = a(\phi_n) = \lim_{z\to\infty} \phi_n(z) - z$ the constant term in the series of ϕ_n. Now $\tilde{\phi}_n = (\phi_n - A_n)/C_n$ is a normalized univalent function, and

$$\tilde{K}_n = \partial\tilde{\phi}_n(\Delta) = \frac{1}{C_n}(K_n - A_n) = \{\frac{z - A_n}{C_n} : z \in K_n\}$$

is a translated and scaled copy of K_n. Because diameter and capacity of connected sets are comparable, this is essentially the same as scaling by diameter.

Let Σ_0 denote the usual space of normalized univalent functions on Δ, that is functions $\varphi(z) = z + O(1/z)$ analytic and univalent in $\{|z| > 1\}$. We equip Σ_0 with the topology of locally uniform convergence. Because Σ_0 is compact, this topology is induced by the metric

$$d(f,g)=\max_{|z|=2}|f(z)-g(z)|.$$

The random process (4.1) induces a measure P_n on Σ_0. This is just the image of the product of (normalized) Lebesgue measure on $\mathbb{T}$ under the map $\sigma_n:(u_1,u_2,...,u_n)\mapsto\tilde{\phi}_n$ from $\mathbb{T}^n$ into Σ_0. Denoting ℓ the Lebesgue measure on $\mathbb{T}$ normalized by $\ell(\mathbb{T})=1$, we thus have $P_n=\ell^n\circ\sigma_n^{-1}$. The following theorem gives a precise version of the statement that the scaling limit of $HL(0)$ exists.

Theorem 1. *There is a probability measure P_∞ on Σ_0 such that the sequence of measures P_n converges weakly to P_∞.*

Remark: Because of the one-to-one correspondence of normalized conformal maps and their image domains (respectively the complement of the image), we can and will view P_∞ as a measure on the space of filled compact sets $\tilde{K}$.
Proof: Consider the "reversal operator" $R_n:\mathbb{T}^n\to\mathbb{T}^n$, defined by $R_n(u_1,u_2,...u_n)=(u_n,u_{n-1},...,u_1)$. Because R_n preserves ℓ^n, we have

$$P_n=\ell^n\circ(\sigma_n\circ R_n)^{-1}.$$

In other words, for the random sets K_n it does not matter in which order we compose the random maps h_n. This is, of course, special to $HL(0)$. We want to show that $\sigma_n\circ R_n$ converges uniformly on $\mathbb{T}^\infty$. To make this more precise, define the map $\tau_n:\mathbb{T}^\infty\to\Sigma_0$ by

$$\tau_n(u_1,u_2,...)=(h_n\circ h_{n-1}\circ...\circ h_1-A_n)/C_n=\sigma_n\circ R_n(u_1,...,u_n)$$

so that

$$P_n=\ell^\infty\circ\tau_n^{-1}.$$

Write $\psi_n=h_n\circ...\circ h_1$ and $f=h_m\circ...\circ h_{n+1}/(1+\lambda)^{(m-n)}$. Because

$$|\phi(z)-c(\phi)z-a(\phi)|\leq\frac{Cc(\phi)}{|z|}$$

for all ϕ univalent in Δ and all $|z| > 1$, we have

$$|\tau_m(u) - \tau_n(u)| = |\frac{f \circ \psi_n - \psi_n}{C_n} - a(\frac{f}{C_n})| \le \frac{C}{C_n|\psi_n|} \le \frac{C}{C_n}.$$

Therefore

$$d(\tau_n(u), \tau_m(u)) \le \frac{C}{C_n}, \quad 1 \le n \le m, \quad u \in \mathbb{T}^\infty$$

and the Theorem follows immediately. □

An alternative description of P_n and P_∞ can be based on the Löwner equation. Consider the Löwner equation (3.2) with driving measure $d\mu_t$ being the Dirac measure at $\xi(t)$,

$$\frac{\partial}{\partial t}\varphi_t(z) = z\varphi_t'(z)\frac{z+\xi(t)}{z-\xi(t)}, \tag{4.2}$$

where $\xi : [a, b] \to \mathbb{T}$ is a piecewise continuous function. The initial value problem (4.2), $\varphi_a(z) = z$ for all $z \in \Delta$, is known to have a unique solution. If we change the initial condition to $\varphi_a(z) = g(z)$ for a univalent function g in Δ, the solution simply is $(g \circ \varphi_t)(z)$. Intuitively, composition of conformal maps corresponds to concatination of the Löwner driving term. More precisely we have

Lemma 4.1. *Let $\xi^{(j)} : [0, b_j] \to \mathbb{T}$ be piecewise continuous, $j = 1, 2$. If $\varphi_t^{(j)}$ are the solutions to (4.2), $\varphi_0^{(j)}(z) = z$, $j = 1, 2$, then $\varphi_{b_1}^{(1)} \circ \varphi_{b_2}^{(2)}$ is the solution (4.2) with $\xi(t) = \xi^{(1)}(t)\chi_{[0,b_1]}(t) + \xi^{(2)}(t - b_1)\chi_{[b_1,b_1+b_2]}(t)$.*

Notice that the conformal map h_δ is the solution to (4.2) with $\xi \equiv 1$ and $t = \log(1 + \lambda) = \tau$. It follows that our function ϕ_n from (4.1) is the solution with

$$\xi(t) = \sum_{k=0}^{n-1} \chi_{[k\tau,(k+1)\tau)} u_k,$$

and therefore the rescaled map $\hat{\phi}_n$ is the solution with

$$\xi(t) = \sum_{k=0}^{n-1} \chi_{[(k-n)\tau,(k-n+1)\tau)} u_k$$

and the initial value $\phi_t(z) = z/C_n$ for $t = -n\tau$. The reason why we have introduced the shift in time becomes apparent below. Let X denote the space of piecewise continuous functions $\xi : (-\infty, 0] \to \mathbb{T}$. With ξ we will associate a conformal map $\varphi_\xi \in \Sigma$ as follows: For $\xi \in X, t_0 < 0$ and every prescribed conformal map g of Δ, the initial value problem (4.2) ($t_0 \le t \le 0$), $\varphi_{t_0} = g$, has a unique solution in Δ. Writing $\varphi_t(z) = \varphi_{t,\xi,g}(z) = a(t)z + ...$, it follows that $\dot{a}(t) = a(t)$ so that $a(t) = a(t_0)e^{t-t_0}$. Assuming that the $z-$coefficient of g is e^{t_0} we thus have the usual normalization $a(t) = e^t$. Using the Koebe distortion theorem and (4.1), it is easy to see that the limit of φ_t as $t_0 \to -\infty$ exists and is independent of the choices of g. Denote this limit $\varphi_{t,\xi}$, then we obtain a map $L : X \to \Sigma$ by setting $L(\xi) = \varphi_{t=0,\xi}$.

Set $t_k = k\log(1+\lambda)$ and consider the subset X_∞ of X consisting of those functions that are constant on the intervals $[t_{k+1}, t_k)$. In other words, the members of X_∞ are the functions

$$\xi(t) = \sum_{k=0}^{\infty} \chi_{[t_{k+1},t_k)} u_k$$

with $u_k \in \mathbb{T}$. The random choice of the u_k induces a probability $\hat{P}$ on X_∞ (this is nothing but ℓ^∞ on $\mathbb{T}^\infty$, of course). We equip both X_∞ and Σ with the (metrizable) topology of uniform convergence on compact subsets. Then it is not hard to see that L is continuous and that the above P_n on Σ_0 converge weakly to the image measure P_∞ of $\hat{P}$ under L. A proof can be based on Theorem 1 and the following observation:

In order to realize our random maps ϕ_n as solutions to the LE over the time interval $(-\infty, 0]$ (instead of the interval $[-n\tau, 0]$ as above), denote $k(z) = z + 1/z$ the conformal map from Δ onto $\mathbb{C} \setminus [-2, 2]$. Notice that k is the solution to (4.2) with $\xi(t) \equiv 1$ for $t \in (-\infty, 0]$. Therefore $k \circ \phi_n$ is the solution to (4.2) with $\xi(t) = \chi_{(-\infty,t_n]} + \sum_{k=0}^{n-1} \chi_{[t_{k+1},t_k)} u_k$, and the rescaling of $k \circ \phi_n - \phi_n$ converges to 0 with $n \to \infty$. We leave the details to the reader.

5 $HL(0)$ clusters are one-dimensional

We will show that the limit clusters of $HL(0)$ are one-dimensional. More precisely, the goal of this section is to prove

Theorem 2. *The measure P_∞ on Σ_0 is supported on the set of those conformal maps ϕ for which $\partial\phi(\Delta)$ has Hausdorff dimension 1, $P_\infty(\dim\partial\phi(\Delta) = 1) = 1$.*

As in Section 4, let us fix $\delta > 0$ and denote by $C_n = (1+\lambda)^n$ the capacity of $K_n = \partial\phi_n(\Delta)$. We will also denote L_n the length of K_n, so that K_0 is the unit circle, $L_0 = l_0 = 2\pi$ and $L_n = l_0 + ... + l_n$ with

$$l_{n+1} = L_{n+1} - L_n = \int_1^{1+\delta} |\phi_n'(r\ u_n)|dr.$$

For a conformal map ϕ of Δ (or $\mathbb{D}$) we will denote

$$\ell(\phi) = \ell(\partial(\phi(\Delta))) = \lim_{r\to 1} \int_0^{2\pi} |\phi'(re^{it})|dt \leq \infty.$$

This is finite if and only if $\partial(\phi(\Delta))$ has finite length (1-dimensional Hausdorff measure). It equals the length if the boundary is a simple closed curve, and in general satisfies

$$\frac{1}{2}\ell(\phi) \leq \text{length } (\partial(\phi(\Delta)) \leq \ell(\phi),$$

see ([6], Section 6.3). In particular we have $L_n \leq \ell(\phi_n)$.

We will begin with the case that δ is large. In this case we even have the stronger result that the limit clusters have finite length, bounded by a constant depending on δ only:

Theorem 3. *There is a constant δ_0 such that for $\delta \geq \delta_0$ and all choices of $u_1, u_2, ..., u_n$,*

$$L_n \leq C(\delta)C_n.$$

Consequently length $\partial(\phi(\Delta)) \leq C(\delta)$ for P_∞-a.e. ϕ.

The proof is based on a modification of the following fact: In a simply connected planar domain $\subset \mathbb{C}$, the length of every hyperbolic geodesic is bounded above by the length of the boundary. Notice that the slit $K_{n+1} - K_n$ is a hyperbolic geodesic in $G = \hat{\mathbb{C}} \setminus K_n$, but that $\infty \in G$ so that we need to use the following variant:

Lemma 5.1. *There is a universal constant C such that*

$$\text{length } \phi([u, 2u]) \leq C\ell(\phi)$$

for all $\phi \in \Sigma$ and all $u \in \mathbb{T}$.

Proof: We will reduce the claim to the case of a geodesic in a domain $\subset \mathbb{C}$. Without loss of generality we may assume $u = 1$. It follows from ([6], Section 9.5) that there is a constant C and a point $v = e^{it}$ with $\pi/2 \leq t \leq \pi$ such that both length $\phi([v, 2v]) \leq C\ell(\phi)$ and length $\phi([\overline{v}, 2\overline{v}]) \leq C\ell(\phi)$. It follows that the image $\phi(S)$ of the sector $S = \{re^{is} : 1 < r < 2, -t < s < t\}$ has boundary of length $\leq C'\ell(\phi)$ and the lemma follow from observing that the arc $\phi[1, 2]$ is a geodesic of $\phi(S)$ (by symmetry). □

The area theorem ([6], Section 1.3) shows that $|\phi'(z)|$ is uniformly bounded in $\{|z| > 2\}$. As an immediate consequence we obtain.

Lemma 5.2. *There is a universal constant C such that*

$$\text{length } \phi([2u, ru]) \leq C(r - 2)$$

for all $\phi \in \Sigma, u \in \mathbb{T}$ and $r \geq 2$. □

Proof of Theorem 3: By Lemmas 5.1 and 5.2 we have for $\delta > 1$

$$L_{n+1} = L_n + \text{length } \phi([u, (1 + \delta)u]) \leq L_n + CL_n + CC_n(\delta - 1)$$

and therefore

$$\frac{L_{n+1}}{C_{n+1}} = \frac{1 + C}{1 + \lambda} \frac{L_n}{C_n} + C'.$$

It follows that $L_n \leq C(\delta)C_n$ if we choose δ_0 so that $\lambda > C$ for $\delta \geq \delta_0$. Hence $\ell(\tilde{\phi}_n)$ is uniformly bounded, and thus the same is true for $\ell(\tilde{\phi})$ for all limit functions. □

We now turn to the case of small δ. We do not know if Theorem 3 still holds. But it is easy to estimate the expected length:

Theorem 4. *There is a constant $C = C(\delta)$ such that*

$$E(L_n) \leq CC_n$$

for all n. Furthermore

$$\limsup_{n\to\infty} \frac{L_n}{C_n\sqrt{\log(C_n)(\log\log(C_n))^{1+\eta}}} = 0$$

almost surely, for all $\eta > 0$.

Proof: From

$$l_{n+1} = \int_1^{1+\delta} |\phi_n'(ru_n)|dr$$

we obtain

$$l_{n+1}^2 \leq \delta \int_1^{1+\delta} |\phi_n'(ru_n)|^2 dr$$

and therefore

$$E(l_{n+1}^2|K_n) \leq \delta \int_{\mathbb{T}} \int_1^{1+\delta} |\phi_n'(ru)|^2 dr|du| = \delta I.$$

Let us define $\psi(z) = 1/\phi_n(1/z)$: ψ is holomorphic and injective in the unit disk sending 0 to 0. Changing variables we get successively

$$I = \int\int_{\frac{1}{1+\delta}\leq|w|\leq 1} \frac{|\psi'(w)|^2}{|\psi(w)|^4}\frac{dwd\overline{w}}{-2i} = \int\int_{\psi(\{\frac{1}{1+\delta}\leq|w|\leq 1\})} \frac{dvd\overline{v}}{-2i|v|^4}.$$

But $\psi'(0) = 1/C_n$ so that, by the Koebe distortion theorem,

$$I \leq \int\int_{|v|\geq\frac{c}{C_n}} \frac{dvd\overline{v}}{-2i|v|^4} \leq CC_n^2.$$

It follows that

$$E(l_{n+1}^2) \leq C\ C_n^2$$

and thus

$$E(l_{n+1}) \leq C\ C_n$$

so that we obtain

$$E(L_n) = E(l_0) + .. + E(l_n) \leq C \sum_{j=0}^{n} C_j = CC_n.$$

We also get that

$$P(\frac{l_{n+1}}{C_n} \geq \sqrt{\log(C_n)(\log\log(C_n))^{1+\eta}}) \leq \frac{C}{n\log(n)^{1+\eta}}.$$

An application of the Borel-Cantelli Lemma yields

$$\limsup_{n\to\infty} \frac{L_n}{C_n\sqrt{\log(C_n)(\log\log(C_n))^{1+\eta}}} < \infty$$

a.s., and the theorem follows. □

Proof of Theorem 2: Fix $\epsilon > 0$ and denote $N(\epsilon, K)$ the minimal number of discs of radius ϵ needed to cover K. Then Theorem 4 shows that

$$E(N(\epsilon, \tilde{K}_n)) \leq \frac{C}{\epsilon}$$

for all n, and we obtain

$$E(N(\epsilon, \tilde{K})) \leq \frac{C}{\epsilon}$$

for the limit clusters of HL(0). Another application of the Borel-Cantelli Lemma easily implies that the Minkowski- and hence the Hausdorff dimension of $\tilde{K}$ is one, almost surely. □

6 Regularized Hastings-Levitov models with small α.

As we have just seen the capacity C_n of K_n grows exponentially when $\alpha = 0$. Our first task is to show that for α small, the capacity still grows fast but not exponentially.

But first we slightly modify the definition of ϵ : Since $C_n \geq 1$ for all n it is clear that $(|z|-1)|\varphi_n'(z)| \geq c > 0$ for $|z| \geq 2$. From now on we fix $\delta < c$ and we define ϵ by the following stopping time construction, as in [4]. We consider the decomposition of the unit circle into dyadic intervals. If I is such an interval with order k we denote by ζ_I its center and $z_I = (1+2^{-k})\zeta_I$. We then define $\epsilon(u) = \epsilon_n(u,\delta) = 2^{-k}$ where k is the order of the minimal dyadic interval containing u and such that

$$2^{-k}|\varphi_n'(z_I)| \geq \delta.$$

Notice that by distortion theorems it remains true that $\epsilon(u)|\varphi'((1+\epsilon)u)| \sim \delta$.

To establish a lower bound for C_n, we combine (3.1) and (3.4) to write

$$\frac{C_{n+1}-C_n}{C_n} \sim \delta^{2-\alpha}\epsilon_n^{\alpha}\,.$$

Using the Hölder inequality with $r > 0$ to be fixed later gives

$$E(\frac{C_{n+1}-C_n}{C_n}|K_n) \sim \delta^{2-\alpha}E(\epsilon_n^{\alpha}) \gtrsim \frac{\delta^{2-\alpha}}{E(\epsilon_n^{-\frac{\alpha}{r}})^r}$$

and we obtain

$$E(\epsilon_n^{-\alpha/r})^r = \left(\sum_{k\geq 0} 2^{k\alpha/r}|E_k|\right)^r$$

where for $k \geq 0$, $E_k = \{u;\ \epsilon_n(u) = 2^{-k}\}$; to estimate the Lebesgue measure of this set we write, using a parameter $p > 0$ to be fixed later

$$\delta^p 2^{kp}|E_k| \sim \int_{E_k} \left|\varphi_n'((1+2^{-k})u)\right|^p |du| \leq \int_{\mathbb{T}} \left|\varphi_n'((1+2^{-k})u\right|^p |du|.$$

In order to estimate this last integral mean, we introduce as before the function

$$\psi(z) = \frac{1}{\varphi(\frac{1}{z})},$$

where we have now dropped the subscript n. We then have, using the obvious change of variable and Koebe inequality

$$\delta^p 2^{kp} |E_k| \leq C C_n^{2p} \int_{\mathbb{T}} \left| \psi'(\frac{u}{1+2^{-k}}) \right|^p |du|.$$

Using then Clunie-Pommerenke theorem (see p.178 of [6]) we get that

$$|E_k| \leq K \delta^{-p} C_n^p 2^{k(\beta(p)-p)}$$

with $\beta(p) = 3p^2 + 7p^3$, the constant K depending only on p. Notice that $\beta(p) = o(p)$ as $p \to 0$, and this is the only thing we will need in the sequel. Combining everything we obtain

$$E(\epsilon_n^{-\frac{\alpha}{r}})^r \leq K \delta^{-pr} C_n^{pr} \left(\sum_{k \geq 0} 2^{-k(p-\beta(p)-\frac{\alpha}{r})} \right)^r. \tag{6.1}$$

In order for this inequality to be meaningful we need that

$$p - \beta(p) - \frac{\alpha}{r} > 0 \Leftrightarrow \tau - \alpha - \frac{\tau\beta(p)}{p} > 0,$$

where τ stands for pr. For any $\tau > \alpha$ we can find (using $\beta(p) = o(p)$) a small $p > 0$ such that

$$\frac{\beta(p)}{p} < \frac{\tau - \alpha}{\tau}$$

and subsequently a (large) value of r such that $pr = \tau$. We have thus proved

Proposition 1. *For every $\eta > 0$ there exists $K = K(\eta)$ such that for $n \geq 0$ and $\alpha > 0$,*

$$E(\frac{C_{n+1} - C_n}{C_n} | K_n) \geq K \delta^{2+\eta} C_n^{-\alpha-\eta}.$$

This proposition implies that at the level of expectations, C_n increases at least as fast as $n^{1/\alpha}$. To obtain an almost sure result we again use a Borel-Cantelli-type argument. First of all we recall

$$E(\epsilon_n^{-\frac{\alpha}{r}}) \leq K(r,p)\delta^{-p}C_n^p,\ r,p > 0.$$

Next we can write, for any $\gamma > 0$ and $\eta > 0$

$$P((C_{n+1}-C_n)C_n^{\alpha+\eta-1} \leq n^{-\gamma}|K_n) = P(\epsilon_n^{-\frac{\alpha}{r}} \geq n^{\frac{\gamma}{r}}\delta^{\frac{2-\alpha}{r}}C_n^{\frac{\alpha+\eta}{r}}|K_n).$$

Setting $\eta = pr - \alpha$, this last quantity is bounded from above by

$$K(r,p)\delta^p C_n^p/(n^{\gamma/r}\delta^{(2-\alpha)/r}C_n^{(\alpha+\eta)/r} \leq K(p,r,\delta)n^{-\gamma/r}$$

. We choose $\eta = \alpha^{3/2},\ r = 5\sqrt{\alpha}$ and $\gamma = 6\sqrt{\alpha}$. We then have $p-\beta(p)-\frac{\alpha}{r} > 0$ and the above probabilities form a convergent series. Now the Borel-Cantelli lemma implies that a.s there exists a constant K such that for all n,

$$(C_{n+1}-C_n)C_n^{\alpha+\eta-1} \geq Kn^{-\gamma}.$$

Integrating these inequalities we obtain

Proposition 2. *If $\alpha < 1$ is small enough, then a.s. there exists $K > 0$ such that for all values of n,*

$$C_n \geq Kn^{\frac{1}{\alpha}(1-7\sqrt{\alpha})}.$$

It is now easy to obtain an estimate of the dimension in this case; it suffices to do so to derive an estimation of L_n in terms of C_n. The estimate proven for $\alpha = 0$ is still valid, i.e.

$$E(l_n|K_n) \leq CC_n,$$

because δ_n is bounded from above by a fixed constant. Fix $\eta > 0$. By Borel-Cantelli, a.s there exists $K > 0$ such that for every n

$$l_n \leq Kn^{1+\eta}C_n \ \Rightarrow\ L_n \leq Kn^{2+\eta}C_n \leq KC_n^{1+(2+\eta)\frac{\alpha}{1-7\sqrt{\alpha}}}.$$

We have thus obtained the

Theorem 5. *For small α and for $\beta > \alpha$, almost surely there exists a constant $K > 0$ such that for every $n \geq 0$,*

$$L_n \leq K C_n^{1+2\frac{\beta}{1-7\sqrt{\beta}}}.$$

In other words the "dimension" of the limit object behaves at most like $1 + 2\alpha$ near $\alpha = 0$. In particular the dimension tends to 1 as α tends to 0.

To end this section, we observe that we can easily obtain a reverse inequality for C_n. Indeed, the Beurling theorem [5] implies that $\epsilon_n \leq C C_n^{-1/2}$ and thus by summation

$$C_n \leq C n^{2/\alpha}.$$

7 The case $\alpha > 1$.

For $n \geq 0$ let us call $N(n)$ the first time that $C_k \geq 2^n$ and $T(n)$ the time necessary to reach $C_k = 2^{n+1}$ from $N(n)$. We have

$$\sum_{k=N(n)}^{N(n)+T(n)} \epsilon_k^\alpha \sim 1.$$

In this expression $\epsilon_k = \epsilon_k(u_1, .., u_k) = \epsilon_k(past, u_k)$. We then use a deep result of Carleson-Makarov [2]:

Theorem 6. *There exists $C, c > 0$ such that*

$$\int_{\mathbb{T}} \epsilon_n(u, \delta)^2 |du| \leq C \left(\frac{\delta}{C_n} \right)^{1+c}.$$

Using this and the fact that $C_k \geq 2^n$ in the time interval considered, we easily get that

$$E(\epsilon_k^\alpha) \leq C \left(\frac{\delta}{2^n} \right)^{\alpha \frac{1+c}{2}}, \ k \in [N(n), N(n) + T(n)].$$

We put $\beta = \alpha\frac{1+c}{2}$. The Bienayme-Chebychev inequality then implies that

$$P(T(n) \leq \frac{2^{n\beta}}{n^2}) \leq P(\epsilon^{\alpha}_{N+1} + .. + \epsilon_{N+[\frac{2^{n\beta}}{n^2}]} \geq \frac{1}{2}) \leq C\frac{2^{n\beta}}{n^2}2^{-n\beta} \leq \frac{C}{n^2}.$$

Now the Borel-Cantelli lemma implies that almost surely $T(n) \geq \frac{2^{n\beta}}{n^2}$ eventually. It follows that the time necessary to reach capacity 2^n is $\geq 2^{n\beta'}$ for n large and $\beta' < \beta$. We deduce the

Theorem 7. *If $\alpha > \frac{2}{1+c}$ there exists $\beta > 1$ such that for $RHL(\alpha)$ almost surely $C_k \leq k^{1/\beta}$ for k large enough.*

As above, the intuitive interpretation of this result is that the dimension of the limiting normalized cluster is $\geq \beta > 1$.

8 The deterministic case.

We again use the Carleson-Makarov regularization, ϵ being the largest $r > 0$ such that

$$\epsilon\left|\varphi'((1+\epsilon)e^{i\theta})\right| = \delta.$$

Lemma 8.1. *If $Log|\varphi'|$ has a small Bloch norm, then ϵ is a Lipschitz function of θ with small norm.*

Recall that the Bloch norm of the holomorphic function $b : \{|z| > 1\} \longrightarrow \mathbb{C}$ is

$$\|b\|_{\mathcal{B}} = \sup\{(|z| - 1)|b'(z)|, |z| > 1\}.$$

If φ is holomorphic and injective in $\{|z| > 1\}$ then the Koebe theorem implies that $\|\log\varphi'\|_{\mathcal{B}} \leq 6$. Conversely, a well-known injectivity criterium says that φ is injective if $\|\log\varphi'\|_{\mathcal{B}} < 1$.

Proof: (Lemma): We simply differentiate with respect to θ the relation

$$\epsilon(\theta)^2\varphi'((1+\epsilon(\theta))e^{i\theta})\overline{\varphi}'((1+\epsilon(\theta))e^{i\theta}) = \delta^2$$

to obtain

$$\epsilon'(\theta)(1+\epsilon(\theta)\mathrm{Re}(\frac{\varphi''}{\varphi'}((1+\epsilon)e^{i\theta})) = (1+\epsilon)\epsilon\mathrm{Im}(\frac{\varphi''}{\varphi'}((1+\epsilon)e^{i\theta})).$$

Now the result follows from the implicit function theorem as soon as $\|\mathrm{Log}\varphi'\|_{\mathcal{B}} < 1$. □

To proceed, we must use the reverse Löwner equation. For a function holomorphic in $\{|z|>1\}$ we will write $Df(z) = zf'(z)$. Up to a multiplicative factor, this coincides with $\partial/\partial\theta$. Recall the equation driving $RHL(\alpha)$:

$$\frac{\partial}{\partial t}\varphi_t = zA_t\varphi_t',\ A_t = \int_{\mathbb{T}} \frac{z+u}{z-u}\epsilon_t(u)^\alpha |du|.$$

We deduce, writing $b_t = \mathrm{Log}\varphi_t'$,

$$\frac{\partial}{\partial t}b_t = DA_t + A_t + A_t Db_t$$

from which it follows that

$$\frac{\partial}{\partial t}(b_t\circ\varphi_t^{-1}) = DA_t\circ\varphi_t^{-1} + A_t\circ\varphi_t^{-1}.$$

The corollary of this study is the following functionnal equation satisfied by b_t:

$$b_t(z) = b_0(\varphi_0^{-1}\circ\varphi_t(z)) + \int_0^t (DA_s + A_s)(\varphi_s^{-1}\circ\varphi_t(z))ds.$$

To exploit this equation we notice that DA_s is the Poisson integral of the function

$$\alpha(\epsilon'\epsilon^{\alpha-1} + H(\epsilon'\epsilon^{\alpha-1}))$$

where ϵ' stands for derivative with respect to θ, while H is the Hilbert transform. We already know that whenever $\|\mathrm{Log}\varphi_t'\|_{\mathcal{B}} < \kappa < 1$ then $\|\epsilon'\|_\infty \le C(\kappa)$. To estimate $\epsilon^{\alpha-1}$ we use the inequality in [6], p.125 to obtain:

Lemma 8.2.

$$c\left(\frac{\delta}{C(t)}\right)^{\frac{1}{1-\kappa}} \leq \epsilon \leq C\left(\frac{\delta}{C(t)}\right)^{\frac{1}{1+\kappa}}.$$

Proof: The quoted inequality gives

$$c\epsilon^{\kappa} \leq \frac{\left|\varphi_t'((1+\epsilon)e^{i\theta})\right|}{C(t)} \leq C\epsilon^{-\kappa}$$

and the lemma follows after multiplication by ϵ. □

We deduce that

$$\epsilon^{\alpha-1} \leq C\left(\frac{C(t)}{\delta}\right)^{\frac{1-\alpha}{1-\kappa}}.$$

Similarly $A_s - 1$ is the Poisson integral of the function

$$(\epsilon^{\alpha}-1) + H(\epsilon^{\alpha}-1)$$

and we have, again whenever $\|\mathrm{Log}\varphi_t'\| < \kappa < 1$,

$$\|\epsilon^{\alpha}-1\|_{\infty} \leq c(\kappa)\alpha\mathrm{Log}\left(\frac{C(t)}{\delta}\right).$$

The preceeding estimates and the boundedness of the Hilbert transform from L^{∞} to BMO imply the

Lemma 8.3. *If* $\|Log\varphi_t'\|_{\mathcal{B}} < \kappa < 1$ *then*

$$\|DA_t + A_t\|_{BM0} \leq C\alpha\left(\frac{C(t)}{\delta}\right)^{\frac{1-\alpha}{1-\kappa}}.$$

For the definition and properties of $BMO(\mathbb{T})$, see [7],[6].

Lemma 8.4. *Whenever* $\|Log\varphi_t'\| < \kappa < 1$ *we have, for* $t \geq 1$ *and* $\alpha' > \alpha$,

$$C\ '(t)C(t)^{\alpha'-1} \geq C\delta^{\alpha'}.$$

Proof: It starts with the identity

$$\frac{C'(t)}{C(t)} = \int \epsilon_t^\alpha(u)|du| :$$

Then the fact that ϵ_t is Lipschitz implies that the reasoning of (6.1) goes through, and the lemma follows. □

We are now ready to state our result. Lets us define

$$M(t) = \sup\{\|\mathrm{Log}\varphi_s'\|_{BMO},\ s \leq t\}.$$

Then, by the John and Nirenberg theorem ([7]), there exists $C, \epsilon_0 > 0$ such that if $M(t) \leq \epsilon_0$ then

$$\forall b \in BMO(\mathbb{T}),\ \forall s \leq t,\ \left\|b \circ (\varphi_s^{-1} \circ \varphi_t)\right\|_{BMO} \leq C\|b\|_{BMO},$$

and $\|\mathrm{Log}\varphi_s'\|_{\mathcal{B}} < \kappa < 1$ for $s \leq t$.

On the other hand, and this is the "raison d'être" of the introduction of BMO in this context, there exists $M > 0,\ 0 < \epsilon_1 < \epsilon_0$ such that if $\|\mathrm{Log}\varphi'\|_{BMO} \leq \epsilon_1$ then $\varphi(\mathbb{T})$ is a Lavrentiev or chord-arc curve (see [6]) with a parameter less than M. In particular it is a rectifiable curve with length bounded from above by $3M\times$diameter. Let us then assume that

$$\|\mathrm{Log}\varphi_0'\|_{BMO} \leq \frac{\epsilon_1}{4C}.$$

Let τ be the first time that $M(\tau) = \epsilon_0$, then

$$\left\|\int_0^\tau (DA_s + A_s) \circ (\varphi_s^{-1} \circ \varphi_t)(.)ds\right\|_{BMO} \leq C\delta^{\frac{-1+\alpha}{1-\kappa}}\alpha\int_0^\tau C(s)^{\frac{1-\alpha}{1-\kappa}}ds.$$

Since

$$\frac{1-\alpha}{1-\kappa} = 1 - \alpha(1-\kappa) + \frac{\kappa}{1-\kappa}$$

and since, by Lemma 8.4

$$C(t)^{1-\frac{\alpha}{1-\kappa}} \leq \delta^{-\frac{\alpha}{1-\kappa}}C'(t),$$

we get that

$$\int_0^\tau C(s)^{\frac{1-\alpha}{1-\kappa}} ds \leq \delta^{-\frac{\alpha}{1-\kappa}} \int_0^\tau C'(s) C(s)^{\frac{\kappa}{1-\kappa}} ds \leq C\delta^{-\frac{\alpha}{1-\kappa}} C(\tau)^{\frac{1}{1-\kappa}}.$$

It follows that

$$C(\tau) \geq c\, \delta \Big(\frac{\epsilon_0}{\alpha}\Big)^{1-\kappa}.$$

We have thus proved the

Theorem 8. *For every $\kappa < 1$ there exists $\eta > 0$ such that if $\|Log\varphi_0'\|_{BMO} < \eta$ and $diam(\varphi_0(\mathbb{T})) \sim 1$ then $\varphi_t(\mathbb{T})$ will stay uniformly chord-arc until its diameter reaches a value of the order of $\delta\alpha^{\kappa-1}$.*

This theorem means that the closer α is to 0 the longer we have to wait until the tongue phenomenon of Hele-Shaw flows appears. This result combined with the dimension estimate for small α in random growth $RHL(\alpha)$ confirms the Hastings-Levitov statement that the flow is steady if $\alpha < 1$.

On the other hand, Carleson-Makarov's result ([2]) easily goes through for α close to 2. This, together with the results above also confirm H-L claim that $RHL(\alpha)$ is turbulent for $\alpha \sim 2$.

However, the present study does not give any hint for what happens if α is close to 1.

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