Loewner Curvature

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Abstract

The purpose of this paper is to interpret the phase transition in the Loewner theory as an analog of the hyperbolic variant of the Schur theorem about curves of bounded curvature. For sufficiently smooth curves γ in a simply connected domain, we will define a family of curves that have a certain conformal self-similarity property, and we will use these curves as comparison curves. Defining their *Loewner curvature* to be constant, we show that every smooth curve has a best-approximating curve of constant Loewner curvature, establish a geometric comparison principle, and show that curves of Loewner curvature bounded by 8 are simple curves.

1 Introduction and Results

A classical theorem of A. Schur [S] says that, among all curves in the plane of curvature bounded above by K, the circular arc of (constant) curvature K minimizes the distance between the endpoints (assuming the length of the curve is smaller than π/K). The analog for the hyperbolic plane was established in [FG]: Again, curves of constant (geodesic) curvature are extremal, but an interesting phenomenon occurs. Namely, if the geodesic curvature (which can simply be defined at the point 0 in the hyperbolic unit disc $\mathbb{D}$ as half of the euclidean curvature, and then at arbitrary points via isometry) is ≤ 1, then the curve is automatically a simple curve that tends to the boundary of $\mathbb{D}$, whereas the curves of constant curvature > 1 are circles in $\mathbb{D}$ in thus do not tend to the boundary.

A similar "phase transition" occurs in the Loewner theory: In the framework of the chordal Loewner equation in the upper half plane (see Section 3.1), if a curve γ has driving term λ which is Hölder continuous with exponent 1/2 and norm $||\lambda||_{1/2} < 4$,

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then γ is a simple curve, whereas for norm ≥ 4 the curve can have self-intersections, see [Li] and [RTZ].

The purpose of this paper is to define a quantity that we call *Loewner curvature*, and to establish some of its properties, particularly regarding the above phenomenon. In Section 2, we will define a family of curves that we call *self-similar*. They are motivated by the Schramm-principle and will be used as comparison curves. These are our curves of constant curvature. In Section 3, we identify these curves using the Loewner differential equation, and obtain that it is a two-parameter family of curves. A one-dimensional collection of these curves is depicted in Figure 1. We also compute the first terms of the series expansion of a curve γ in terms of the first and second derivative of the Loewner driving term, and show in Proposition 3.3 that

$$\gamma(t) = 2i\sqrt{t} + a\,t - i\frac{a^2}{8}\,t^{3/2} + b\,t^2 + o(t^2)$$

for t near 0, where $a = \frac{2}{3}\lambda'(0)$ and $b = \frac{4}{15}\lambda''(0) + \frac{1}{135}\lambda'(0)^3$. This allows us (Proposition 3.4) to associate with every sufficient smooth curve a best-fitting comparison curve, and leads to our formula for Loewner curvature

$$LC_\gamma(t) = \frac{\lambda'(t)^3}{\lambda''(t)}$$

in Section 4. There we also show (4.4) that γ is a simple curve if $LC_\gamma(t) < 8$. The constant 8 is best possible and corresponds to the constant 4 in the criterion $||\lambda||_{1/2} < 4$ for simple Loewner traces. In Section 5, we establish a comparison principle (Theorem 5.1) by showing that a bound on Loewner curvature implies that the curve stays to one side of the corresponding curve of constant curvature. In particular, Loewner curvature could be alternatively defined by geometric comparison with constant curvature curves.

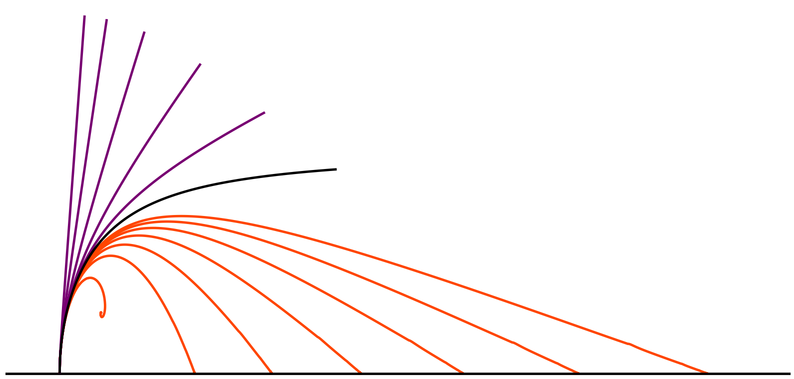

Figure 1: Some curves of constant Loewner curvature. In our normalization, the purple curves have negative curvature, the black curve has infinite curvature, and the red curves have positive curvature.

2 A family of curves

In this section, we will introduce a family of curves, which we call "self-similar curves." In Section 4, we will define the Loewner curvature for these curves to be constant. The Loewner curvature for an arbitrary curve will be defined by comparison to this family. In order to give some motivation for our definition of the self-similar curves, we will first remind the reader of Schramm's principle and then formulate a deterministic version.

Let Ω be a Jordan domain with two distinct marked boundary points, a and b. For each triple (Ω, a, b), we assume that there is a family $\Gamma_{\Omega,a,b}$ of simple curves in $\overline{\Omega}$ that begin at a and a probability measure $\mu_{\Omega,a,b}$ on $\Gamma_{\Omega,a,b}$. (Note: This could be made slightly more general: we do not need simple curves, but we must require that the curves do not cross over themselves.) The measures $\mu_{\Omega,a,b}$ are said to satisfy Schramm's principle if they satisfy the following:

* Conformal invariance: Given a conformal map ϕ of the domain Ω, then

  $$\phi(\mu_{\Omega,a,b}) = \mu_{\phi(\Omega),\phi(a),\phi(b)}.$$

  (The measure $\phi(\mu_{\Omega,a,b})$ on the family of curves $\phi(\Gamma_{\Omega,a,b})$ is the push-forward of the measure $\mu_{\Omega,a,b}$.)

* Domain Markov property: Let $\gamma[0,t]$ be the initial part of a curve in $\Gamma_{\Omega,a,b}$. Let $\mu_{\Omega,a,b\,|\gamma[0,t]}$ be the conditional probability measure given $\gamma[0,t]$. Then

  $$\mu_{\Omega,a,b\,|\gamma[0,t]} = \mu_{\Omega\setminus\gamma[0,t],\gamma(t),b}$$

  or in other words, given the initial part of the curve $\gamma[0,t]$, the conditional measure is the same as the measure on curves starting at $\gamma(t)$ in the slit domain $\Omega \setminus \gamma[0,t]$.

Given $\gamma[0,t]$, define G_t to be the set of all conformal maps $\phi_t : \Omega \setminus \gamma[0,t] \to \Omega$ with $\phi_t(\gamma(t)) = a$ and $\phi_t(b) = b$. Thus if $\mu_{\Omega,a,b}$ satisfy conformal invariance and the domain Markov property, then

$$\phi_t\left(\mu_{\Omega,a,b\,|\gamma[0,t]}\right) = \mu_{\Omega,a,b}$$

for any $\phi_t \in G_t$. This means that the measure is invariant under "mapping down" the initial part of a curve.

This leads us to a deterministic Schramm's principle, that is, a deterministic axiom that will characterize an important family of curves. In this paper we will be interested in curves in a domain that begin at one of two marked boundary points. Let us introduce our notation:

Definition 2.1. *Given a Jordan domain Ω with distinct boundary points a and b and given $T \in (0,\infty]$, the notation $\gamma : (0,T) \to (\Omega, a, b)$ means that $\gamma[0,T)$ is a simple (deterministic) curve with $\gamma(0,T) \subset \Omega$ and $\gamma(0) = a$.*

For ease of notation, we will often simply write γ for $\gamma(0,T)$, the image of the open time interval. We note that $\gamma(T)$ may be defined, in which case it is possible that $\gamma(T) \in \partial\Omega$ or $\gamma(T) \in \gamma(0,T)$. We want to understand the curves that satisfy the following self-similarity property:

Definition 2.2. *A curve $\gamma : (0,T) \to (\Omega, a, b)$ is self-similar if $\gamma \in C^3$ and if for each $t \in (0,T)$, there exists a conformal map $\phi_t \in G_t$ so that*

$$\phi_t(\gamma(t,T)) = \gamma.$$

We write $S(\Omega, a, b)$ for the family of self-similar curves in the marked domain (Ω, a, b).

In other words, self-similar curves are invariant under "mapping down" any initial part of the curve, modulo a conformal renormalization. In Theorem 3.2 below, we will determine all self-similar curves. In particular, we obtain the following:

Corollary 2.3. *Given a Jordan domain Ω with distinct boundary points a and b, there exists a unique two-parameter family of self-similar curves.*

The assumption on the smoothness of self-similar curves in Definition 2.2 is probably unnecessary and only used to establish the regularity needed for our proof of Theorem 3.2. It could be replaced by requiring that the ϕ_t are continuous in t.

3 Self-similar curves and the Loewner framework

If we know $S(\Omega_0, a_0, b_0)$ for one fixed triple (Ω_0, a_0, b_0), then we can obtain $S(\Omega, a, b)$ via a conformal map $\psi : \Omega_0 \to \Omega$ with $\psi(a_0) = a$ and $\psi(b_0) = b$. That is,

$$S(\Omega, a, b) = \{\psi \circ \gamma \, : \, \gamma \in S(\Omega_0, a_0, b_0)\}.$$

It is most convenient to work with $\Omega_0 = \mathbb{H}$, the upper half-plane, with $a_0 = 0$ and $b_0 = \infty$, and for the remainder of this section we will work in this setting. We take the viewpoint that doing computations with the triple $(\mathbb{H}, 0, \infty)$ is like "working in coordinates."

Our goal is to use the chordal Loewner equation to understand the family $S(\mathbb{H}, 0, \infty)$. We begin with a brief review of the chordal Loewner equation (that the expert can safely skip.)

3.1 The chordal Loewner equation

Let $\lambda : [0,T] \to \mathbb{R}$ be continuous, and let $z \in \mathbb{H}$. Then the chordal Loewner differential equation is the following initial value problem:

$$\begin{aligned} \frac{\partial}{\partial t} g_t(z) &= \frac{2}{g_t(z) - \lambda(t)}, \\ g_0(z) &= z. \end{aligned} \tag{3.1}$$

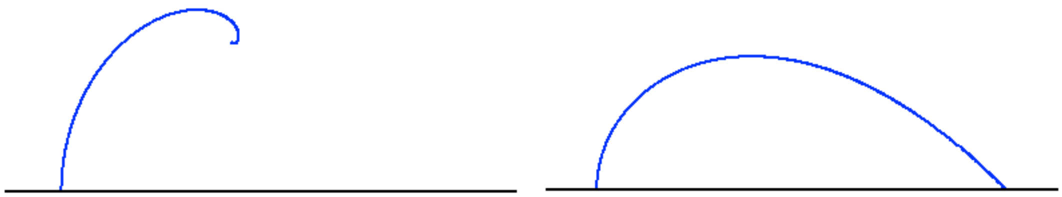

Figure 2: The curves generated by driving function $c - c\sqrt{1-t}$ when $c = 3$ and $c = 5$.

We call λ the driving function, since it determines the unique solution $g_t(z)$, which is guaranteed to exist for some time interval. The only obstacle in solving (3.1) is obtaining a zero in the denominator. We collect these problem points together in the set K_t, called the (Loewner) hull. So

$$K_t := \{z \in \mathbb{H} \, : \, g_s(z) = \lambda(s) \text{ for some } s \in (0, t]\}.$$

If $z \notin K_t$, then $g_t(z)$ is well-defined, and it can be shown that $\mathbb{H} \backslash K_t$ is simply connected and g_t is the unique conformal map from $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$ that satisfies the following normalization at infinity (called the hydrodynamic normalization):

$$g_t(z) = z + \frac{c(t)}{z} + O(\frac{1}{z^2}). \tag{3.2}$$

Further, $c(t) = 2t$, and this quantity is called the halfplane capacity of K_t. For more details, see [La].

In the simplest situation, there is a simple curve γ so that $K_t = \gamma(0, t]$. It is also possible that K_t is the complement of the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$ for a non-simple curve γ that touches itself or $\mathbb{R}$. In both these situations, the curve γ is called the trace. For example, if $\lambda(t) = 3 - 3\sqrt{1-t}$, then K_1 is the simple curve shown on the left of Figure 2. When $\lambda(t) = 5 - 5\sqrt{1-t}$, then K_1 is the union of the curve shown in the right of Figure 2 and the region under the curve. See [KNK] for a detailed discussion of these examples. There are other possibilities for K_t (such as a space-filling curve), but these do not concern us in this paper.

The Loewner equation provides a correspondence between continuous driving functions and certain families of hulls. Above, we briefly explained how one can start with a continuous function and use the Loewner equation to obtain a hull. On the other hand, if we have an appropriate family of hulls[1], we can determine the driving function. We will assume for simplicity that the family of hulls is $K_t = \gamma(0, T]$ for a simple curve γ in $\mathbb{H}$ with $\gamma(0) \in \mathbb{R}$. Let $g_t : \mathbb{H} \backslash \gamma(0, t] \to \mathbb{H}$ be the conformal map with the hydrodynamic normalization at infinity (the normalization given in (3.2).) We can reparameterize γ as necessary so that in (3.2) $c(t) = 2t$, in which case we say that γ is parametrized by halfplane capacity. Then one can show that the conformal maps g_t satisfy (3.1) with $\lambda(t) = g_t(\gamma(t))$, that is, λ is the conformal image of the tip of the curve γ.

[1] For the precise statement, see Section 4.1 in [La].

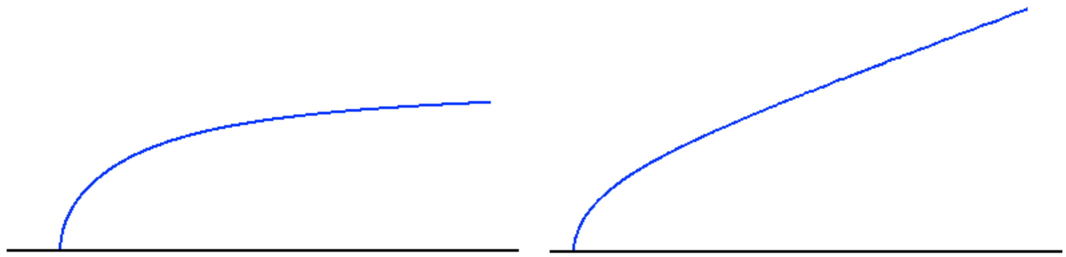

Figure 3: The curves generated by driving functions t and $5\sqrt{1+t}-5$.

We list some simple but useful properties of the chordal Loewner equation. Assume that the hulls K_t are generated by the driving term $\lambda(t)$.

1. *Scaling*: For $r>0$, the driving term of the scaled hulls rK_{t/r^2} is $r\lambda(t/r^2)$.
2. *Translation*: For $x\in\mathbb{R}$, the driving term of shifted hulls K_t+x is $\lambda(t)+x$.
3. *Reflection*: The driving term of the reflected hulls $R_I(K_t)$ is $-\lambda(t)$, where R_I denotes reflection in the imaginary axis.
4. *Concatenation*: For fixed τ, $\lambda(\tau+t)$ is the driving function of the mapped hulls $g_\tau(K_{\tau+t})$.

We end this section by mentioning the following theorem, which follows from the work in [EE] and [M]:

Theorem 3.1 (Earle, Epstein, Marshall)**.** *Assume $\gamma:(0,T)\to(\mathbb{H},0,\infty)$ has driving function $\lambda(t)$. If any parametrization of γ is C^n, then the halfplane-capacity parametrization of γ is in $C^{n-1}(0,\tau)$ and $\lambda\in C^{n-1}(0,\tau)$, where τ is the halfplane-capacity of γ (and may be infinite.)*

3.2 Computations in coordinates

The following theorem implies Corollary 2.3.

Theorem 3.2. *Assume $\gamma:(0,T)\to(\mathbb{H},0,\infty)$ has driving function $\lambda(t)$. Then $\gamma\in S(\mathbb{H},0,\infty)$ if and only if $\lambda(t)$ is one of following driving functions:*

$$0,\quad ct,\quad c\sqrt{\tau}-c\sqrt{\tau-t},\quad \textit{or}\quad c\sqrt{\tau+t}-c\sqrt{\tau} \tag{3.3}$$

where $c\in\mathbb{R}\setminus\{0\}$ and $\tau>0$. Further λ is completely determined by the two real parameters $\lambda'(0)$ and $\lambda''(0)$.

Before the proof, we wish to briefly describe the curves generated by the driving functions listed in (3.3).

1. If $\lambda(t)=0$, then γ will be the ray $\{ir : r>0\}$.

2. If $\lambda(t) = ct$, then γ will be a scaled version of the left curve shown in Figure 3 (and reflected around the imaginary axis if $c < 0$). This curve goes to infinity, but remains within a bounded distance of the real line.

3. If $\lambda(t) = c\sqrt{\tau} - c\sqrt{\tau - t}$ and $|c| < 4$, then γ will end with an infinite spiral. The spiral is so tight as to not be visually discernable, as in the example shown in Figure 2. If $\lambda(t) = c\sqrt{\tau} - c\sqrt{\tau - t}$ and $|c| \geq 4$, then $\gamma(\tau) \in \mathbb{R}$, and the angle of intersection depends on c. These examples were first described and implicitely computed in [KNK]. In both cases, changing τ scales the picture.

4. If $\lambda(t) = c\sqrt{\tau + t} - c\sqrt{\tau}$, then γ will approach infinity asymptotic to a ray, as shown in the right-hand example in Figure 3. The angle of the ray depends on c, and changing τ scales the picture.

Remark: The notion of self-similarity introduced in this paper is a generalization of the notion introduced in [LMR], where we defined a curve γ in $\mathbb{H}$ with finite half-plane capacity to be self-similar if $g_t(\gamma(t,T))$ is a translation and dilation of γ for all $t \in (0,T)$. Proposition 3.1 in [LMR] states that γ is self-similar (under the more restrictive definition) if and only if the driving function is $\lambda(t) = k + c\sqrt{\tau - t}$ for some $c, k \in \mathbb{R}$ and $\tau > 0$. [LMR] also contains geometric constructions for these curves, which gives a conceptual and simple way to obtain the solutions to (3.1) with driving function $\lambda(t) = k + c\sqrt{\tau - t}$.

Now for the proof of Theorem 3.2:

Proof. Assume that $\gamma \in S(\mathbb{H}, 0, \infty)$. Without loss of generality, we assume γ is parametrized by halfplane capacity. This means that for each $t \in (0,T)$, there exists a conformal map $\phi_t : \mathbb{H} \setminus \gamma[0,t] \to \mathbb{H}$ with $\phi_t(\gamma(t)) = 0$ and $\phi_t(\infty) = \infty$ so that $\phi_t(\gamma(t,T)) = \gamma$. It follows that $\phi_t = r_t \cdot (g_t - \lambda(t))$, where $r_t > 0$ and g_t is the solution to (3.1). Thus

$$r_t \cdot (g_t(\gamma(t,T)) - \lambda(t)) = \gamma.$$

By the concatenation, scaling and translation properties,

$$r_t \cdot \left(\lambda\left(t + \frac{s}{r_t^2}\right) - \lambda(t)\right) = \lambda(s). \tag{3.4}$$

Theorem 3.1 implies that $\lambda \in C^2(0,T)$. Since the left-hand side of (3.4) is twice differentiable for $s = 0$, the same must be true for $\lambda(s)$, and so $\lambda \in C^2[0,T)$. By taking derivatives in (3.4) with respect to s and setting $s = 0$,

$$\frac{1}{r_t}\lambda'(t) = \lambda'(0) \quad \text{and} \quad \frac{1}{r_t^3}\lambda''(t) = \lambda''(0). \tag{3.5}$$

If $\lambda'(0) = 0$, then $\lambda'(t) = 0$ for all t, and hence $\lambda(t) = 0$. If $\lambda''(0) = 0$, then $\lambda''(t) = 0$ for all t and $\lambda(t) = ct$ for $c \in \mathbb{R}$. If $\lambda'(0) \neq 0$ and $\lambda''(0) \neq 0$, then $\lambda'(t)$ and $\lambda''(t)$ are nonzero for all t, and (3.5) gives

$$\frac{\lambda'(t)^3}{\lambda''(t)} = \frac{\lambda'(0)^3}{\lambda''(0)}. \tag{3.6}$$

Setting $A = \frac{\lambda'(0)^3}{\lambda''(0)}$ and solving (3.6) gives

$$\lambda'(t) = \pm\left(B - \frac{2}{A}t\right)^{-1/2} \quad \text{and} \quad \lambda(t) = \pm A\sqrt{B - \frac{2}{A}t} \mp A\sqrt{B}.$$

Thus, there exists $c \neq 0$ and $\tau > 0$ so that $\lambda(t) = c\sqrt{\tau} - c\sqrt{\tau - t}$ (when $A > 0$) or $\lambda(t) = c\sqrt{\tau + t} - c\sqrt{\tau}$ (when $A < 0$.)

Conversely, if λ is one of the driving functions $0, ct$, $c\sqrt{\tau} - c\sqrt{\tau - t}$, or $c\sqrt{\tau + t} - c\sqrt{\tau}$, then λ will satisfy (3.4), and this implies that $\gamma \in S(\mathbb{H}, 0, \infty)$. $\square$

Proposition 3.3. *Assume that $\lambda \in C^2[0,T)$ is the driving function for a curve γ. Further, we assume that $\lambda(0) = 0$ and either $\lambda'(0) \neq 0$ or both $\lambda'(0)$ and $\lambda''(0)$ are 0. Then in the halfplane-capacity parametrization, γ satisfies*

$$\gamma(t) = 2i\sqrt{t} + a\,t - i\frac{a^2}{8}\,t^{3/2} + b\,t^2 + o(t^2) \tag{3.7}$$

for t near 0, where $a = \frac{2}{3}\lambda'(0)$ and $b = \frac{4}{15}\lambda''(0) + \frac{1}{135}\lambda'(0)^3$.

Proof. This proof needs two components: computations and Theorem 3.3 in [W]. For the computations, we need to show that the curves generated by driving functions 0, ct, $c\sqrt{\tau} - c\sqrt{\tau - t}$, and $c\sqrt{\tau + t} - c\sqrt{\tau}$ all satisfy (3.7). These computations are straightforward. As done in [KNK], one can solve (3.1) for the driving functions in our list, obtaining an implicit equation for $\gamma(t)$. Then we simply expand in a power series to obtain (3.7).

For the second step, choose $\lambda_*(t)$ to be the driving function in this list that satisfies $\lambda'(0) = \lambda_*'(0)$ and $\lambda''(0) = \lambda_*''(0)$. Let γ_* be the curve generated by λ_*. Then Theorem 3.3 in [W] implies that for $t \in [0, \epsilon]$,

$$|\gamma(t) - \gamma_*(t)| \leq C \sup_{0 \leq t \leq \epsilon} |\lambda(t) - \lambda_*(t)|$$

for small enough ϵ. This in turn implies (3.7).

$\square$

The previous proposition shows that the two parameters $\lambda'(0)$ and $\lambda''(0)$ determine the Loewner trace up to 4th order in $t^{1/2}$ (excluding the case $\lambda'(0) = 0$ and $\lambda''(0) \neq 0$.) These two parameters also uniquely determine a self-similar curve in $S(\mathbb{H}, 0, \infty)$ by Theorem 3.2. As we see in the next proposition, this allows us to match a given curve with a unique "best-fitting" self-similar curve. We say $\gamma_* \in S(\Omega, a, b)$ is the best-fitting curve to $\gamma : (0, T) \to (\Omega, a, b)$ at a if under the conformal transformation $\psi : (\Omega, a, b) \to (\mathbb{H}, 0, \infty)$, the halfplane-capacity parametrization of $\psi(\gamma_*)$ and $\psi(\gamma)$ match up to 4th order (or satisfy $\lambda'(0) = \lambda_*'(0) = 0$.)

Proposition 3.4. *Let $\gamma : (0, T) \to (\Omega, a, b)$ be C^3. At each point $\gamma(t)$, the curve $\gamma(t, T)$ has a unique best-fitting curve $\gamma_* \in S(\Omega \setminus \gamma[0, t], \gamma(t), b)$.*

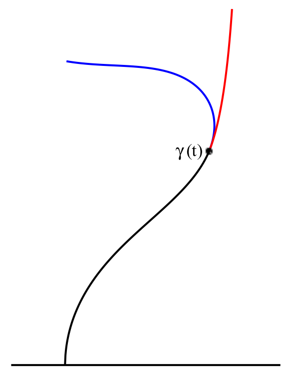

Figure 4: Illustration of Proposition 3.4 : $\gamma(0,t)$ shown in black, $\gamma(t,T)$ in blue and γ_* in red.

Proof. By conformal transformation, it suffices to consider the situation when $\Omega = \mathbb{H}$, $a = 0$ and $b = \infty$. By Theorem 3.1, since γ is C^3, we can deduce that both λ and the halfplane-capacity parametrization of γ are in $C^2(0,\tau)$ (where τ is the halfplane capacity of γ and may be infinite.) For the rest of the proof, we assume γ is parametrized by halfplane capacity and $T = \tau$.

For $t \in (0,T)$, $\lambda'(t)$ and $\lambda''(t)$ are defined. If $\lambda'(t) = 0$, we set $\gamma_*(t) = 2i\sqrt{t}$. Otherwise, we choose the unique $\gamma_* \in S(\mathbb{H},0,\infty)$ satisfying $\lambda'(t) = \lambda'_*(0)$ and $\lambda''(t) = \lambda''_*(0)$. Then by Proposition 3.3 and Theorem 3.2, γ_* is the unique best-fitting curve to $g_t(\gamma(t,T)) - \lambda(t)$ at the origin. Thus $g_t^{-1}(\gamma_*) \in S(\mathbb{H} \setminus \gamma[0,t], \gamma(t), \infty)$ is the unique best-fitting curve to $\gamma(t,T)$ at the point $\gamma(t)$.

□

4 Loewner curvature

4.1 The definition of Loewner curvature

We are now ready to give the definition of Loewner curvature, notated $LC_\gamma(t)$. We begin with assigning constant curvature for $\gamma \in S(\mathbb{H},0,\infty)$ as follows:

Definition 4.1. *Let $\gamma \in S(\mathbb{H},0,\infty)$. Then LC_γ, the Loewner curvature of γ, is defined to be the following constant:*

1. *If γ is generated by 0, then $LC_\gamma \equiv 0$.*
2. *If γ is generated by ct, then $LC_\gamma \equiv \infty$.*
3. *If γ is generated by $c\sqrt{\tau} - c\sqrt{\tau - t}$, then $LC_\gamma \equiv c^2/2$.*
4. *If γ is generated by $c\sqrt{\tau + t} - c\sqrt{\tau}$, then $LC_\gamma \equiv -c^2/2$.*

Note that for $\gamma \in S(\mathbb{H},0,\infty)$, the Loewner curvature is scaling invariant by definition. For the self-similar curves $\gamma \in S(\Omega,a,b)$, $LC_\gamma(t)$ is defined to satisfy conformal invariance. For a C^3 curve $\gamma : (0,T) \to (\Omega, a, b)$, we define $LC_\gamma(t)$ by comparison to the curves of constant curvature as follow:

Definition 4.2. *Let $\gamma : (0,T) \to (\Omega, a, b)$ be C^3. The Loewner curvature of γ at the point $\gamma(t)$, notated $LC_\gamma(t)$, is defined to be LC_{γ_*}, where $\gamma_* \in S(\Omega \setminus \gamma(0,t], \gamma(t), b)$ is the unique best-fitting curve to $\gamma(t,T)$ at $\gamma(t)$.*

Proposition 3.4 (illustrated in Figure 4) guarantees that γ_* exists.

Although our approach to defining Loewner curvature via comparison curves is natural, it is difficult to use this definition to compute $LC_\gamma(t)$. The following lemma shows that computations are straightforward in the Loewner framework. (The nice formula that appears in this lemma also explains the particular definition for LC_γ in cases 3 and 4 of Definition 4.1.)

Proposition 4.3. *Let $\gamma : (0,T) \to (\mathbb{H}, 0, \infty)$ be a C^3 curve that is parametrized by halfplane capacity, and let λ be the corresponding driving function. Then for $t \in (0,T)$,*

$$LC_\gamma(t) = \frac{\lambda'(t)^3}{\lambda''(t)}. \tag{4.1}$$

Since $LC_\gamma(t) = 0$ precisely when $\lambda'(t) = 0$, if the right-hand side of (4.1) is in the indeterminate form "$\frac{0}{0}$", we declare it to equal 0.

Proof. By Definition 4.1, equation (4.1) is true for $\gamma \in S(\mathbb{H}, 0, \infty)$. For $t \in (0,T)$, let $\gamma_* \in S(\mathbb{H} \setminus \gamma(0,t], \gamma(t), \infty)$ be the unique best-fitting curve to $\gamma(t,T)$ at $\gamma(t)$. Suppose that λ_* generates $g_t(\gamma_*) - \lambda(t)$, which is the unique best-fitting curve in $S(\mathbb{H}, 0, \infty)$ to $g_t(\gamma(t,T)) - \lambda(t)$ at 0. Then

$$LC_\gamma(t) = LC_{\gamma_*} = \frac{\lambda_*'(0)^3}{\lambda_*''(0)} = \frac{\lambda'(t)^3}{\lambda''(t)}.$$

The last equality is due to Proposition 3.3.

□

The next theorem says that if the curvature is small, then γ cannot "curve" enough to hit itself or the boundary (except at the marked point b.)

Theorem 4.4. *Let $\gamma : (0,T) \to (\Omega, a, b)$ be C^3. If $LC_\gamma(t) < 8$ for all $t \in (0,T)$, then $\gamma(0,T]$ is a simple curve in $\Omega \cup \{b\}$.*

The constant 8 is best possible, as the curve in case 3 of Definition 4.1 with $c = 4$ shows. The following lemma is key component of the proof.

Lemma 4.5. *Let $\gamma : (0,T) \to (\mathbb{H}, 0, \infty)$ be C^3 with driving function λ. If $0 < LC_\gamma(t) \leq A$, then λ is Lip(1/2) with norm bounded above by $\sqrt{2A}$.*

Proof. Since γ is C^3, Theorem 3.1 implies that $\lambda \in C^2(0,T)$. By Proposition 4.3, $0 < \lambda'(t)^3/\lambda''(t) \leq A$ for all $t \in (0,T)$. Without loss of generality, we assume that T is finite, and we will show that $|\lambda(t) - \lambda(s)| \leq \sqrt{2A}\sqrt{|t-s|}$ for $t, s \in [0,T]$.

Set $\sigma = \lambda'$, and assume that $\sigma > 0$ (if not, replace σ with $-\sigma$.) Then

$$0 < \sigma^3 \le A\sigma'. \tag{4.2}$$

The solution to $\sigma^3 = A\sigma'$ is

$$\sigma_{A,B}(t) = \sqrt{A/2}(B-t)^{-1/2}$$

for some $B > 0$. Assume for the moment that $\sigma \le \sigma_{A,T}$, and let $0 \le s < t \le T$. Then

$$\lambda(t) - \lambda(s) = \int_s^t \sigma(u)\,du \le \int_s^t \sigma_{A,T}(u)\,du \le \sqrt{2A}\sqrt{t-s}.$$

It remains to show $\sigma(t) \le \sigma_{A,T}(t)$ for all $t \in (0,T)$. Suppose to the contrary that there is a time $s_0 < T$ so that $\sigma(s_0) > \sigma_{A,T}(s_0)$. This will imply, as we show in the remainder of the proof, that there is a time $\tau < T$ with $\sigma(\tau) = \infty$, contradicting the fact that $\sigma \in C^1(0,T)$.

For simplicity, assume $s_0 = 0$ (by shifting time if necessary), and set

$$\alpha = \sigma(0)/\sigma_{A,T}(0) > 1. \tag{4.3}$$

Let N be large enough so that

$$N^2 \sum_{n=N}^{\infty} \frac{1}{n^3} < \frac{\alpha^2}{2}.$$

This is possible since $\alpha > 1$ and the limit as $N \to \infty$ of the left hand side is $1/2$. Recursively define an increasing sequence $\{s_n\}$ by

$$s_{n+1} = s_n + \frac{2T}{\alpha^2}\frac{N^2}{(N+n)^3}.$$

Finally set $\tau = \lim_{n\to\infty} s_n$, which satisfies

$$\tau = \sum_{n=0}^{\infty}(s_{n+1}-s_n) = \frac{2T}{\alpha^2}N^2\sum_{n=0}^{\infty}\frac{1}{(N+n)^3} < T.$$

We will show by induction that

$$\sigma(s_n) \ge \alpha\sqrt{\frac{A}{2T}}\,\frac{N+n}{N}. \tag{4.4}$$

Rewriting (4.3) gives $\sigma(0) = \alpha\sqrt{A/2T}$, which is the base case. Assume that (4.4) holds for some fixed n. By (4.2), σ is increasing and $\sigma' \ge \sigma^3/A$. Thus,

$$\begin{aligned}
\sigma(s_{n+1}) &= \sigma(s_n) + \int_{s_n}^{s_{n+1}} \sigma'(t)\,dt \\
&\ge \sigma(s_n) + \frac{1}{A}\sigma(s_n)^3\ (s_{n+1}-s_n) \\
&\ge \alpha\sqrt{\frac{A}{2T}}\,\frac{N+n}{N}\left(1 + \frac{1}{A}\left(\alpha\sqrt{\frac{A}{2T}}\,\frac{N+n}{N}\right)^2 \frac{2T}{\alpha^2}\frac{N^2}{(N+n)^3}\right) \\
&= \alpha\sqrt{\frac{A}{2T}}\frac{N+n+1}{N}.
\end{aligned}$$

Thus (4.4) is true for all n, showing that $\sigma(\tau) = \infty$. $\square$

Now the proof of Theorem 4.4:

Proof. It suffices to prove Theorem 4.4 for the triple $(\mathbb{H}, 0, \infty)$. Let λ be the driving function for γ, in which case Proposition 4.3 implies that $LC_\gamma(t) = \lambda'(t)^3/\lambda''(t)$. Thus $\lambda'(t)^3/\lambda''(t) \leq 8$ for all $t \in (0, T)$.

We split this proof into two cases. For the first case, suppose that there exists a time t_0 so that $\lambda'(t_0)^3/\lambda''(t_0) > 0$. We claim that $\lambda'(t)^3/\lambda''(t) \in (0, \infty)$ for all $t \in (t_0, T)$. If this claim is not true, then there must be a first time $t_1 \in (t_0, T)$ so that $\lambda'(t_1)^3/\lambda''(t_1) \notin (0, \infty)$. This implies that either $\lambda'(t_1) = 0$ or $\lambda''(t_1) = 0$. However, for $t \in [t_0, t_1)$, $\lambda'(t)$ and $\lambda''(t)$ must have the same sign, implying that $0 < |\lambda'(t_0)|^3 \leq |\lambda'(t)|^3 \leq 8|\lambda''(t)|$. Hence, $\lambda'(t)$ and $\lambda''(t)$ are bounded away from zero on $[t_0, t_1]$, proving the claim. Now in this case, Lemma 4.5 implies that λ is Lip(1/2) on $[t_0, T]$ with norm strictly less than 4. Thus by Theorem 2 in [Li], $g_{t_0}(\gamma(t_0, T])$ is a simple curve in $\mathbb{H} \cup \{\infty\}$. Therefore $\gamma(0, T]$ is simple in $\mathbb{H} \cup \{\infty\}$.

For the next case, suppose that $\lambda'(t)^3/\lambda''(t) \in [-\infty, 0]$ for all $t \in (0, T)$. This means that λ' and λ'' always have the opposite sign (or could be 0). Thus, for $\epsilon > 0$, λ' is bounded on $[\epsilon, T)$, which implies that λ is locally Lip(1/2) with small norm. In particular, we can find an interval (t_0, T) so that λ is Lip(1/2) on (t_0, T) with norm strictly less than 4. We finish the proof as in the first case.

$\square$

For $\gamma : (0, T) \to (\mathbb{H}, 0, \infty)$, Loewner curvature is invariant under scaling. This follows from Definitions 4.1 and 4.2 and can also be verified computationally by Proposition 4.3. To distinguish between different curves with the same Loewner curvature, we must choose a "scale" which we define as follows:

Definition 4.6. *Assume $\gamma : (0, T) \to (\mathbb{H}, 0, \infty)$ has driving function $\lambda \in C^2[0, T)$ with $\lambda'(0) \neq 0$. Then the scale of γ at 0 is $a = \frac{2}{3}\lambda'(0)$, which is the coefficient of the linear term in* (3.7).

The scale of γ at 0 is simply a multiple of the Euclidean curvature of γ at 0. To see this, let γ_r be the upper half-circle with radius r, which is driven by $3r - 3\sqrt{2}\sqrt{r^2/2 - t}$. Thus, Proposition 3.3 implies that

$$\gamma_r(t) = 2i\sqrt{t} + \frac{2}{r}t + O(t^{3/2}).$$

Comparing this to (3.7) shows that the half-circle that best matches the curve γ at its base has radius $r = 2/a = 3/\lambda'(0)$. In other words, the scale of γ at 0 is twice the Euclidean curvature of γ at its base.

4.2 Loewner curvature is neither local, nor reversible

Although Loewner curvature shares some similarities with other notions of curvature, there are significant differences. For instance, Loewner curvature is not reversible (but

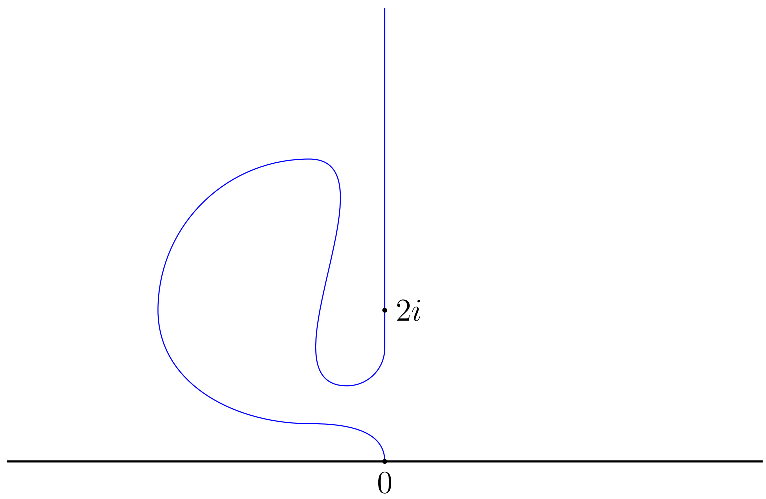

Figure 5: A curve for which Loewner curvature is not local and not reversible.

depends on the orientation of the curve), and it is not local (but depends on the "past" of the curve). To see why this must be true, let's assume for the moment that Loewner curvature is a purely local concept, and consider the smooth curve γ shown in Figure 5. Let t_0 be the time that $\gamma(t_0) = 2i$. If Loewner curvature were purely local, then $LC_\gamma(t) = 0$ for $t > t_0$, since locally near $\gamma(t)$ the curve γ looks like the vertical slit, which has constant Loewner curvature 0. This means that $g_{t_0}(\gamma)$ must be a vertical ray, and therefore $\gamma(t_0, \infty)$ would be a hyperbolic geodesic in $\mathbb{H} \setminus \gamma[0, t_0]$, which it is not. This example also shows that Loewner curvature cannot be reversible: in contrast to the forward-direction, if we traverse γ from ∞ to 0, then the Loewner curvature will be 0 prior to reaching $2i$.

4.3 Existence of a curve with given Loewner curvature

It is natural to ask the following question:

Question: Given a continuous function $l : [0, T] \to \mathbb{R} \cup \{\infty\}$, does there exist a curve $\gamma : (0, T) \to (\mathbb{H}, 0, \infty)$ with $LC_\gamma(t) = l(t)$?

We will need to revise this question before we can answer it in Theorem 4.7 below. To understand the needed revision, we consider two examples. First suppose l is a nonzero constant function. Then there are an infinite number of self-similar curves with Loewner curvature equal to l. Thus we should have the freedom to specify another parameter, for instance, the scale (or equivalently, the Euclidian curvature) of the curve at 0. However, if we are in the case that the curve is driven by $\lambda(t) = c\sqrt{\tau} - c\sqrt{\tau - t}$, then specifying the scale determines the halfplane capacity τ of the curve. There is a difficulty if τ is smaller than the desired value of T. Since $\lambda'(t) \to \infty$ as $t \to \tau$, it is impossible to continue λ past τ so that λ remains C^2.

For our second example, suppose $l(t) = (1-t)^2$ for $t \in [0,2]$, and let $a \neq 0$ be the scale of the curve. Now we wish to solve $\lambda'(t)^3/\lambda''(t) = (1-t)^2$ with $\lambda'(0) = 3a/2$. Separation of variables leads to

$$\lambda'(t)^{-2} = \frac{4}{9\,a^2} - \frac{2t}{1-t}.$$

Regardless of the value of a, $|\lambda'(t)| \to \infty$ before time 1. These examples show that the best we can hope for is the existence of a curve on a small time interval.

Theorem 4.7. *Let $l : [0,T] \to \mathbb{R} \cup \{\infty\}$ be a continuous function with $l(0) \neq 0$, and let $a \in \mathbb{R} \setminus \{0\}$. Then there exists $\tau > 0$ and a curve $\gamma : (0,\tau) \to (\mathbb{H}, 0, \infty)$ so that $LC_\gamma(t) = l(t)$ for $t \in (0,\tau)$ and γ has scale a at 0.*

Proof. Since $l(0) \neq 0$, there is some interval $[0,\tau]$ so that $l(0)$ is bounded away from zero. Then we can solve solve

$$\sigma(t) = l(t)\sigma'(t), \qquad \sigma(0) = 3a/2$$

by separation of variables to obtain

$$\sigma(t)^{-2} = \frac{4}{9\,a^2} - 2\int_0^t \frac{1}{l(s)}\,ds.$$

By taking a smaller value of τ if needed, we can ensure that the right-hand side is positive and bounded away from zero for all $t \in [0,\tau]$. Therefore, we can integrate $\sigma(t)$ to obtain a function $\lambda(t)$ defined on $[0,\tau]$. Now let γ be the curve generated by λ via the Loewner equation. Proposition 4.3 implies that $LC_\gamma(t) = l(t)$, and Proposition 3.3 implies that the scale of γ at 0 is a.

□

5 A comparison principle

In Theorem 4.4, a bound on the Loewner curvature yielded global information about a curve. The next theorem is in the same spirit. Let us first set the stage, by introducing the normalized curves of constant curvature, γ^c and Γ^c. For $c > 0$, set

$$\lambda_c(t) = c^2 - c\sqrt{c^2 - t} \quad \text{and} \quad \Lambda_c(t) = c\sqrt{c^2 + t} - c^2.$$

Let γ^c be the curve generated by λ_c, and similarly, let Γ^c denote the curve generated by Λ_c. Each of these curves has the same scale, since $\lambda_c'(0) = 1/2 = \Lambda_c'(0)$. See Figures 6 and 7.

We list some additional terminology and notation that is needed:

* For $c \geq 4$, define $\tau_c = c^2$, and for $c < 4$, define τ_c to be the first time that the tangent vector $\gamma_c'(t)$ points downward. We will often use γ for $\gamma[0,T]$ and γ^c for $\gamma^c[0,\tau_c]$.

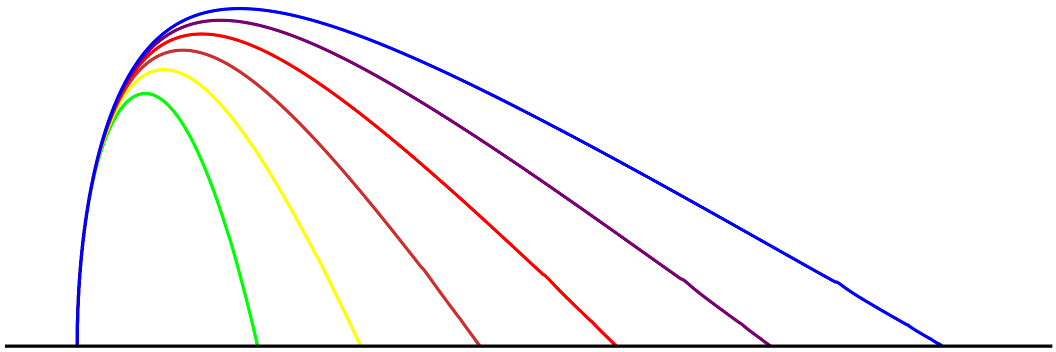

Figure 6: Curves γ^c generated by $c^2 - c\sqrt{c^2 - t}$ for $c = 5, 6, 7, 8, 9, 10$.

* The phrase "the base of γ_1 is to the right of the base of γ_2" means that for γ_1 and γ_2 parametrized by height h, $\mathrm{Re}\,(\gamma_1(h)) \geq \mathrm{Re}\,(\gamma_2(h))$ for small h.
* The phrase "γ_1 is below γ_2" means that the base of γ_1 is to the right of the base of γ_2 and the curves γ_1 and γ_2 never cross (although we allow them to touch.)

Theorem 5.1. *Assume γ is generated by $\lambda \in C^2[0, T)$ with $\lambda(0) = 0$ and $\lambda'(0) = 1/2$. Let $c > 0$.*

1. *If $0 < LC_\gamma(t) \leq c^2/2$, then $\gamma[0, T)$ is below $\gamma^c[0, \tau_c]$.*
2. *If $c^2/2 \leq LC_\gamma(t) < \infty$, then $\gamma^c[0, \tau_c]$ is below the curve $\gamma[0, T)$.*
3. *If $-\infty < LC_\gamma(t) \leq -c^2/2$, then $\gamma[0, T)$ is below $\Gamma^c[0, \infty)$.*
4. *If $-c^2/2 \leq LC_\gamma(t) < 0$, then $\Gamma^c[0, \infty)$ is below $\gamma[0, T)$.*

The proof of each part of this theorem has two steps. First we must analyze the base of γ using the power series expansion given in Proposition 3.3. For the second step, we analyze the curve away from its base by comparing the flow under λ and the flow under λ_c (or Λ_c). This involves changing time for one of the flows to allow easier comparison.

For g_t and γ generated by λ, set

$$\gamma_t = g_t(\gamma) - \lambda(t),$$

that is, we "map down" $\gamma[t, T]$ by the function g_t and then shift so that the curve is rooted at the origin. We also have the notation g_t^c, γ^c, and γ_t^c for the corresponding functions and curves associated with the driving function λ_c. We wish to compare γ_t and γ_t^c. However, we want to compare curves that are the same scale. This leads us to introduce the following time change: for $t \in [0, T]$, set

$$s = s(t) = (\lambda_c')^{-1}(\lambda'(t)).$$

In particular, we will have that $\lambda_c'(s) = \lambda'(t)$.

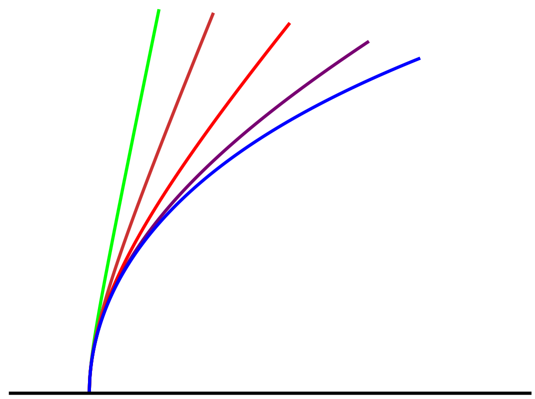

Figure 7: Curves Γ^c generated by $c\sqrt{c^2+t}-c^2$ for $c = 1/2, 1, 2, 4, 8$.

Lemma 5.2. *Assume $\lambda \in C^2[0,T)$ with $\lambda'(0) = 1/2$ and $\lambda(0) = 0$, and let $c > 0$. For $t \in [0,T]$, set $s = s(t) = (\lambda_c')^{-1}(\lambda'(t))$.*

1. *If $0 < LC_\gamma(t) \le c^2/2$, then, $s'(t) \ge 1$.*
2. *If $c^2/2 \le LC_\gamma(t) < \infty$, then, $s'(t) \le 1$.*
3. *If $-\infty < LC_\gamma(t) \le -c^2/2$, then $s'(t) \ge 1$.*
4. *If $-c^2/2 \le LC_\gamma(t) < 0$, then $s'(t) \le 1$.*

Proof. We assume $0 < LC_\gamma(t) \le c^2/2$ to prove the first statement. The other statements are proved in the same manner.

The reparametrization function $s(t)$ is well-defined because $\lambda_c'(t)$ is strictly increasing from $1/2$ to ∞ and λ is strictly increasing with $\lambda'(0) = 1/2$. Since $\lambda_c'(s) = \lambda'(t)$ and

$$\frac{\lambda'(t)^3}{\lambda''(t)} \le \frac{c^2}{2} = \frac{\lambda_c'(s)^3}{\lambda_c''(s)},$$

it must be true that $\lambda_c''(s) \le \lambda''(t)$. Thus $s'(t) = \dfrac{\lambda''(t)}{\lambda_c''(s)} \ge 1$. $\square$

Now for the proof of Theorem 5.1:

Proof. We assume that $0 < LC_\gamma(t) \le c^2/2$, and we will prove the first statement; the remaining statements are proved in a similar manner.

The first step is to prove that the base of γ^c is to the left of the base of γ. Since $0 < LC_\gamma(t) \le c^2/2$, then

$$\frac{\lambda'(0)^3}{\lambda''(0)} \le \frac{c^2}{2} = \frac{\lambda_c'(0)^3}{\lambda_c''(0)}.$$

Thus $\lambda''(0) \ge \lambda_c''(0)$, since $\lambda'(0) = \lambda_c'(0) = 1/2$. Proposition 3.3 implies that

$$\gamma(t) = 2i\sqrt{t} + \frac{1}{2}t - i\frac{1}{32}t^{3/2} + \left(\frac{4}{15}\lambda''(0) + \frac{1}{1080}\right)t^2 + o(t^2)$$

for t near 0. Fix $h > 0$, and let t and t_c satisfy $\operatorname{Im}\gamma(t) = h = \operatorname{Im}\gamma^c(t_c)$. That is,

$$2\sqrt{t} - \frac{1}{32}t^{3/2} + o(t^2) = h = 2\sqrt{t_c} - \frac{1}{32}t_c^{3/2} + o(t_c^2),$$

which implies that

$$t - t_c = o\left((t \vee t_c)^2\right).$$

Then

$$\begin{aligned}\operatorname{Re}\gamma(t) - \operatorname{Re}\gamma^c(t_c) &= [\operatorname{Re}\gamma(t_c) - \operatorname{Re}\gamma^c(t_c)] - [\operatorname{Re}\gamma(t_c) - \operatorname{Re}\gamma(t)] \\ &= \frac{4}{15}\left(\lambda''(0) - \lambda_c''(0)\right)t_c^2 + o\left((t \vee t_c)^2\right)\end{aligned}$$

So for h small enough, $\operatorname{Re}\gamma(t) - \operatorname{Re}\gamma^c(t_c) > 0$, proving that the base of the curve γ^c is to the left of the base of the curve γ.

Since γ^c is self-similar, γ_t^c is simply a scaled version of γ^c, that is $\gamma_t^c = r \cdot \gamma^c$. By Proposition 3.3, the scale of γ_t^c is $2\lambda_c'(t)/3$, and the scale of $r \cdot \gamma^c$ is $(2\lambda_c'(0)/3) \cdot (1/r)$. Thus $r = \lambda_c'(0)/\lambda_c'(t)$, and

$$\gamma_t^c = \frac{1}{2\lambda_c'(t)} \cdot \gamma^c.$$

Since $\lambda_c'(t)$ is strictly increasing, γ_t^c is "shrinking" as time increases.

Suppose that γ is not below γ_c. Then γ and γ^c must cross, and there exists $z_0 = \gamma(\tau)$, where $0 < \tau < T$, so that z_0 is "outside" γ^c. If $c \geq 4$, z_0 is outside γ^c when z_0 is in the unbounded component of $\mathbb{H} \setminus \gamma^c$. If $c < 4$, z_0 is outside γ^c, if there is a continuous curve β joining $\gamma^c(\tau_c)$ to $\mathbb{R}\setminus\{0\}$ in $\mathbb{H}\setminus(\gamma^c \cup \gamma(0,\tau))$ with z_0 in the unbounded component of $\mathbb{H} \setminus (\gamma^c \cup \beta)$. Set $z_t = g_t(z_0) - \lambda(t)$ and set $w_t = g_{s(t)}^c(w_0) - \lambda_c(s(t))$ for a point $w_0 \in \mathbb{H}$ which will be specified later. Then

$$\partial_t\, z_t = \frac{2}{z_t} - \lambda'(t) \quad \text{and} \quad \partial_t\, w_t = \left(\frac{2}{w_t} - \lambda'(t)\right) \cdot s'(t) \tag{5.1}$$

using the fact that $\lambda_c'(s(t)) = \lambda'(t)$. Since $z_0 = \gamma(\tau)$, then $z_t \in \gamma_t$ for $0 < t < \tau$.

We claim that z_t is outside $\gamma_{s(t)}^c$ for all $t < T$. Let t_0 be a time that z_{t_0} is outside of $\gamma_{s(t_0)}^c$, and choose $w_0 \in \mathbb{H}$ so that $z_{t_0} = w_{t_0}$. Now $w_t \notin \gamma_{s(t)}^c$ for $t > t_0$. By (5.1) and Lemma 5.2, the direction of motion for z_t and w_t at time $t = t_0$ is the same, but $|\partial_t w_{t_0}| \geq |\partial_t z_{t_0}|$. Because of the shape of the curve γ^c, in order for z_t to move below $\gamma_{s(t)}^c$, z_t must either move faster than w_t or in a different direction than w_t (or both.)

Thus z_t remains outside $\gamma_{s(t)}^c$ for all $t < T$. However, this leads to the contradiction. The kill-time for z_0 is τ and $\tau < T$. However, the base of γ_t always remains below (i.e. not outside) $\gamma_{s(t)}^c$, and so z_0 cannot be killed before time T.

□

To round out this section, we mention the following result which is related to the infinite curvature case:

Proposition 5.3. *Assume γ is generated by $\lambda \in C^1[0,T)$ with $\lambda(0) = 0$, and let $c > 0$.*

1. *If $\lambda'(t) \geq c$, then γ is below the curve generated by ct.*

2. If $\lambda'(t) \leq c$, then the curve generated by ct is below γ.

This can be proved in the same manner as Theorem 5.1.

References

[EE] C. Earle, A. Epstein, Quasiconformal variation of slit domains, *Proc. Amer. Math. Soc.* **129** (2001), 3363–3372.

[FG] J.L. Fernández, A. Granados, On geodesic curvature and conformal mapping, *St. Petersburg Math. J.* **9** (1998), 615–637.

[KNK] W. Kager, B, Nienhuis, L. Kadanoff, Exact solutions for Loewner Evolutions, *J. Statist. Phys.* **115** (2004), 805–822.

[La] G. Lawler, Conformally invariant processes in the plane, *Mathematical Surveys and Monographs,* **114**, American Mathematical Society, Providence, RI, 2005.

[Li] J. Lind, A sharp condition for the Loewner equation to generate slits, *Ann. Acad. Sci. Fenn. Math.* **30** (2005), 143–158.

[LMR] J. Lind, D.E. Marshall, S. Rohde, Collisions and Spirals of Loewner Traces, *Duke Math. J.* **154** (2010), 527–573.

[M] D.E. Marshall, Derivation of Chordal Loewner from Radial Loewner, preprint.

[RTZ] S. Rohde, H. Tran, M. Zinsmeister, The Loewner equation and Lipschitz graphs, preprint.

[S] A. Schur, Über die Schwarz'sche Extremaleigenschaft des Kreises unter den Kurven konstanter Krümmung, *Math. Ann.***83**(1921), 143–148.

[W] C. Wong, Smoothess of Loewner slits, *Trans. Amer. Math. Soc.* **366** (2014), 1475–1496.