Conformal Invariance and Probability

S. Rohde, UW

At the intersection of probability, statistical physics and complex analysis, the Schramm-Loewner Evolution SLE has created quite a stir by providing rigorous proofs to several outstanding conjectures involving Brownian motion, percolation, and several other discrete mathematical lattice models (see Figure 1 for illustrations of some of the random curves we will encounter, and Figure 2 for some deterministice curves related to the Loewner equation). The aim of this two-quarter course is to give a self-contained introduction to SLE.

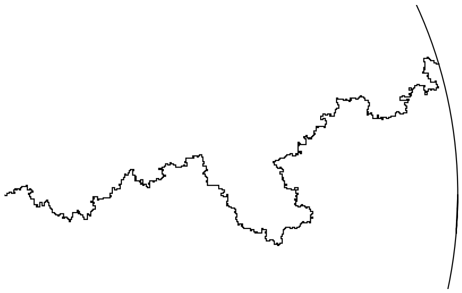
Topics covered include:

1. The self-avoiding walk
2. Basic theory of conformal maps: Riemann mapping theorem, distortion theorems, the zipper algorithm
3. The Loewner differential equation (chordal, radial, planar)
4. Basic theory of Brownian Motion
5. SLE, Schramm's principle
6. Basic Stochastic Calculus (Ito Integral, diffusions, Dynkin's formula; applications: Conformal invariance of BM, recurrence vs transience, area, first phase transition of SLE)
7. Path properties of the deterministic and the stochastic LE (Continuity, Phases, Transience, Dimensions)
8. SLE_6 (restriction property; conformally invariant measures) and Cardy's formula
9. Smirnov's Theorem (convergence of critical percolation interfaces $\to SLE_6$)
10. $SLE_{8/3}$ (restriction; SAW)
11. Intersection exponents for BM, Mandelbrot conjecture, and the work of Lawler, Schramm and Werner

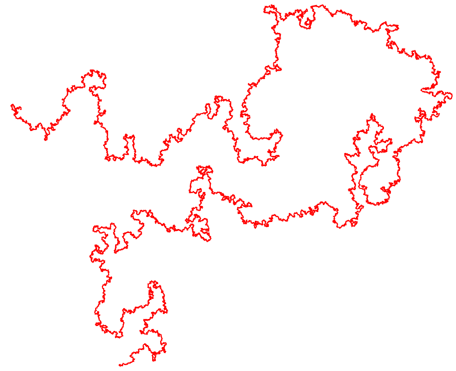
We will draw from several sources, largely from Greg Lawlers book [L], drafts of course notes of Zhenqing Chen and myself [CR] and of Michel Zinsmeister [Z] (both available by email upon request). The following books are on reserve in the library:

[GM] John Garnett and Don Marshall, Harmonic Measure
[L] Greg Lawler, Conformally Invariant Processes in the Plane
[P1] Christian Pommerenke, Univalent Functions
[P2] Christian Pommerenke, Boundary Behaviour of Conformal Maps
[MP] Peter Moerters and Yuval Peres, Brownian Motion
[O] Bernt Oksendal, Stochastic Differential Equations
[KS] Ioannis Karatzas and Steven Shreve, Brownian Motion and Stochastic Calculus

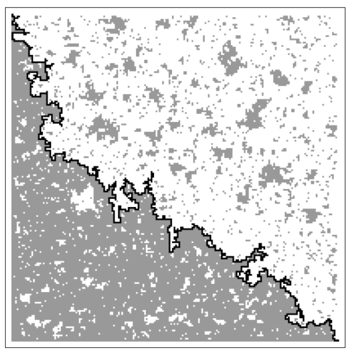
Prerequisites: Graduate level Real- and Complex Analysis, and basic (undergraduate) probability. Advanced probability will make it easier to appreciate and understand some details, but the course is designed to assume no prior knowledge of Brownian motion or stochastic calculus.



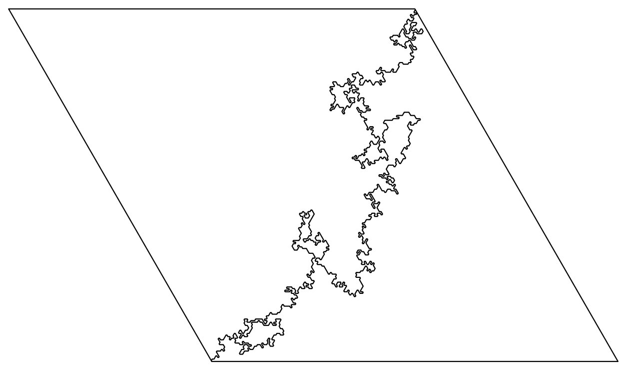
(a) Loop Erased Random Walk, $\kappa = 2$



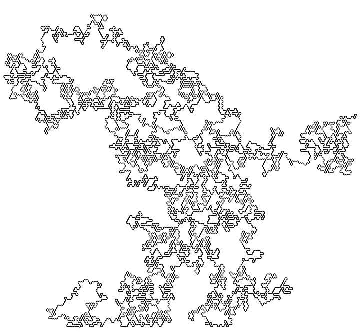
(b) Self Avoiding Walk, $\kappa = \frac{8}{3}$?



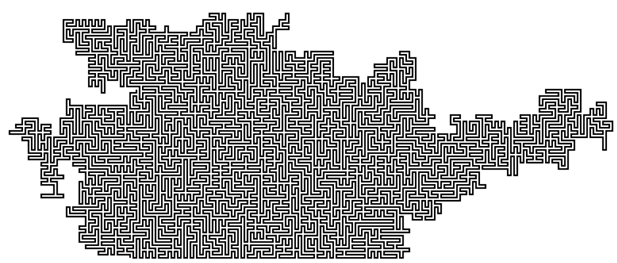
(c) Critical Ising interface, $\kappa = 3$



(d) Harmonic explorer, $\kappa = 4$



(e) Percolation, $\kappa = 6$



(f) Uniform Spanning Tree, $\kappa = 8$

Figure 1: Random curves converging to SLE_κ's for various values of κ

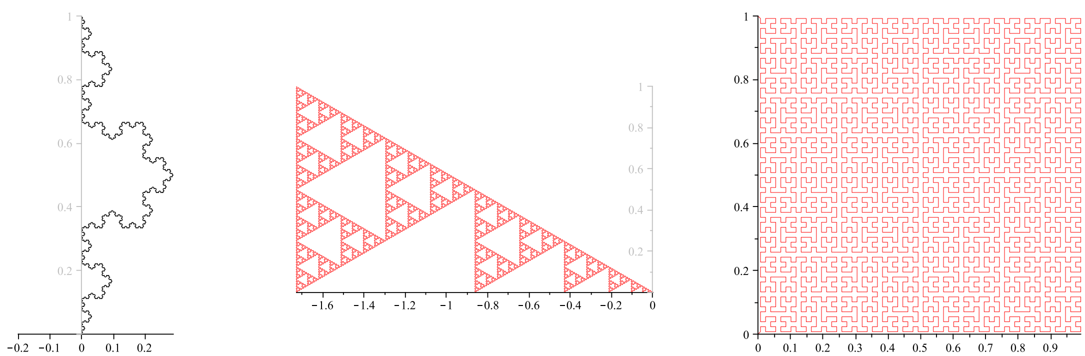


Figure 2: Three curves that can be desribed with the Loewner equation.