University of Washington
Conformal Invariance and Probability
S. Rohde Autumn 2011

Exercise Set 2

Problem 1. If f is analytic in the unit disc $\mathbb{D}$, show that f is Hölder continuous with exponent $0 < \alpha < 1$ if and only if there is C such that

$$|f'(z)| \leq C(1-|z|)^{1-\alpha} \quad \text{for all} \quad z \in \mathbb{D}.$$

Problem 2. Show that the hulls of the (chordal) Loewner equation driven by the continuous driving term $\lambda : [0,T]$ are generated by a continuous curve if there is a function $\phi : (0,1] \to (0,\infty)$ such that $\phi(0+) = 0$ and

$$\int_0^{\varepsilon} |g_t^{-1}\,'(\lambda(t)+iy)| \leq \phi(\varepsilon)$$

for all t.

Problem 3. Denote ℓ the *euclidean* length of a curve, and diam the euclidean diameter of a set.

a) If $f : \mathbb{D} \to \mathbb{C}$ is a conformal map and if $A \subset \mathbb{D}$ is any compact connected set that intersects both lines $\{x = 0\}$ and $\{x = 1/2\}$, show that

$$\ell(f([0,1/2])) \leq C_1 \operatorname{diam} f([0,1/2]) \leq C_2 \operatorname{diam} f(A)$$

(with constants $C_{1,2}$ independent of f and A).

b) Use a) to show the "Gehring-Hayman inequality"

$$\ell(\sigma) \leq C \operatorname{length}(\alpha)$$

for any simply connected domain $D \subset \mathbb{C}$, any curve $\alpha \subset D$, and the hyperbolic geodesic σ with the same endpoints as α.

Problem 4. Apply the Feynman-Kac formula to obtain a PDE satisfied by

$$u(t,x,y) = E[|f_t'(x+iy)|^p],$$

where f_t is backward SLE_κ.

Problem 5. With $f_t(z)$ backward SLE_κ, write $f_t(z) - \sqrt{\kappa}B_t = X_t + iY_t$ and $u = u(t,z) = \log Y_t$. Also write $f_u(z) = f_{t(u,z)}(z)$ and set

$$G(u,x,y) = E[|f_u'(x+iy)|^p].$$

Show that $G(u,rx,ry) = G(u,x,y)$ for each $r > 0$.

Due date : Monday, October 31.