University of Washington
Conformal Invariance and Probability - Math 583
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Exercise Set 3

Problem 1. Let $G \subset \mathbb{C}$ be a domain for which each boundary component is an analytic curve, let $z_0 \in G$ and let $g(z) = g(z, z_0)$ be Green's function of G with pole at z_0. Use Green's formula

$$\int_D ((\Delta u)v - u\Delta v)dxdy = \int_{\partial D} (\frac{\partial u}{\partial n}v - u\frac{\partial v}{\partial n})ds$$

to show that the harmonic measure of G at z_0 is given by

$$d\omega = -\frac{1}{2\pi}\frac{\partial g}{\partial n}$$

where n is the outward pointing normal.

Problem 2. For $0 < x < 1$ compute $\omega(-x, \mathbb{D} \setminus [0,1], \partial\mathbb{D})$ and show that this is $< c\sqrt{x}$.

Problem 3. Let E be a connected compact subset of $\overline{\mathbb{H}}$ with $0 \in E$ and diameter d less than 1. Show that there are two constants c_1, c_2 independent of E such that $c_1 d \leq \omega(i, \mathbb{H} \setminus E, E) \leq c_2 d$.

Problem 4. Let $K \subset \mathbb{H}$ be bounded, relatively closed and such that $\mathbb{H} \setminus K$ is simply connected. Let ϕ be a conformal map from $\mathbb{H} \setminus K$ onto $\mathbb{H}$. Denote I_- and I_+ the two unbounded components of $\mathbb{R} \setminus \overline{K}$, and J the complement $\mathbb{R} \setminus (I_- \cup I_+)$. Further denote $K' = K \cup J \cup K^*$, where $*$ denotes reflection in $\mathbb{R}$. Use Schwarz reflection to show that ϕ has an analytic extension to $\overline{\mathbb{C}} \setminus K'$ (in particular to a neighborhood of ∞), and use this to show the existence of the hydrodynamically normalized conformal map $g_K : \mathbb{H} \setminus K \to \mathbb{H}$. Show that g_K extends to a conformal bijection $\overline{\mathbb{C}} \setminus K' \to \overline{\mathbb{C}} \setminus I$, where $I \subset \mathbb{R}$ is the smallest interval with the property that $g_K(z) \to I$ as $z \to K$ (more precisely, I is smallest with the property that whenever $z_n \in \mathbb{H}$, $z_n \to w$ with $w \in K$, then every limit point of $g_K(z_n)$ belongs to I). Conclude that the *logarithmic* capacity of K' and I are equal.

Due date : Monday, May 9.