Chapter 3

Problems

1. $P\{6 \mid \text{different}\} = P\{6, \text{different}\}/P\{\text{different}\}$

$$= \frac{P\{\text{1st} = 6, \text{2nd} \neq 6\} + P\{\text{1st} \neq 6, \text{2nd} = 6\}}{5/6}$$

$$= \frac{2\ 1/6\ 5/6}{5.6} = 1/3$$

could also have been solved by using reduced sample space—for given that outcomes differ it is the same as asking for the probability that 6 is chosen when 2 of the numbers 1, 2, 3, 4, 5, 6 are randomly chosen.

2. $P\{6 \mid \text{sum of } 7\} = P\{(6, 1)\}/1/6 = 1/6$

$P\{6 \mid \text{sum of } 8\} = P\{(6, 2)\}/5/36 = 1/5$

$P\{6 \mid \text{sum of } 9\} = P\{(6, 3)\}/4/36 = 1/4$

$P\{6 \mid \text{sum of } 10\} = P\{(6, 4)\}/3/36 = 1/3$

$P\{6 \mid \text{sum of } 11\} = P\{(6, 5)\}/2/36 = 1/2$

$P\{6 \mid \text{sum of } 12\} = 1.$

3. $P\{E \text{ has } 3 \mid N - S \text{ has } 8\} = \dfrac{P\{E \text{ has } 3, N - S \text{ has } 8\}}{P\{N - S \text{ has } 8\}}$

$$= \frac{\binom{13}{8}\binom{39}{18}\binom{5}{3}\binom{21}{10}\Big/\binom{52}{26}\binom{26}{13}}{\binom{13}{8}\binom{39}{18}\Big/\binom{52}{26}} = .339$$

4. $P\{\text{at least one } 6 \mid \text{sum of } 12\} = 1$. Otherwise twice the probability given in Problem 2.

5. $\dfrac{6}{15}\dfrac{5}{14}\dfrac{9}{13}\dfrac{8}{12}$

6. In both cases the one black ball is equally likely to be in either of the 4 positions. Hence the answer is 1/2.

7. $P1\ g \text{ and } 1\ b \mid \text{at least one } b\} = \dfrac{1/2}{3/4} = 2/3$

8. 1/2

9. $$P\{A = w \mid 2w\} = \frac{P\{A = w, 2w\}}{P\{2w\}}$$

$$= \frac{P\{A = w, B = w, C \neq w\} + P\{A = w, B \neq w, C = w\}}{P\{2w\}}$$

$$= \frac{\frac{1}{3}\frac{2}{3}\frac{3}{4} + \frac{1}{3}\frac{1}{3}\frac{1}{4}}{\frac{1}{2}\frac{2}{3}\frac{3}{4} + \frac{1}{3}\frac{1}{3}\frac{1}{4} + \frac{2}{3}\frac{2}{3}\frac{1}{4}} = \frac{7}{11}$$

10. 11/50

11. (a) $$P(B \mid A_s) = \frac{P(BA_s)}{P(A_s)} = \frac{\frac{1}{52}\frac{3}{21} + \frac{3}{52}\frac{1}{51}}{\frac{2}{52}} = \frac{1}{17}$$

Which could have been seen by noting that, given the ace of spades is chosen, the other card is equally likely to be any of the remaining 51 cards, of which 3 are aces.

(b) $$P(B \mid A) = \frac{P(B)}{P(A)} = \frac{\frac{4}{52}\frac{3}{51}}{1 - \frac{48}{52}\frac{47}{51}} = \frac{1}{33}$$

12. (a) $(.9)(.8)(.7) = .504$

(b) Let F_i denote the event that she failed the ith exam.

$$P(F_2 \mid (F_1^c F_2^c F_3^c)^c) = \frac{P(F_1^c F_2)}{1 - .504} = \frac{(.9)(.2)}{.496} = .3629$$

13. $$P(E_1) = \binom{4}{1}\binom{48}{12}\Big/\binom{52}{13}, \qquad P(E_2 \mid E_1) = \binom{3}{1}\binom{36}{12}\Big/\binom{39}{13}$$

$$P(E_3 \mid E_1E_2) = \binom{2}{1}\binom{24}{12}\Big/\binom{26}{13}, \qquad P(E_4 \mid E_1E_2E_3) = 1.$$

Hence,

$$p = \binom{4}{1}\binom{48}{12}\Big/\binom{52}{13} \cdot \binom{3}{1}\binom{36}{12}\Big/\binom{39}{13} \cdot \binom{2}{1}\binom{24}{12}\Big/\binom{26}{13}$$

14. $$\frac{5}{12}\frac{7}{14}\frac{7}{16}\frac{9}{18} - \frac{35}{768}.$$

15. Let E be the event that a randomly chosen pregnant women has an ectopic pregnancy and S the event that the chosen person is a smoker. Then the problem states that

$$P(E|S) = 2P(E|S^c),\ P(S) = .32$$

Hence,

$$\begin{aligned} P(S|E) &= P(SE)/P(E) \\ &= \frac{P(E|S)P(S)}{P(E|S)P(S) + P(E|S^c)P(S^C)} \\ &= \frac{2P(S)}{2P(S) + P(S^c)} \\ &= 32/66 \approx .4548 \end{aligned}$$

16. With S being survival and C being C section of a randomly chosen delivery, we have that

$$\begin{aligned} .98 = P(S) &= P(S|C).15 + P(S|C^2)\ .85 \\ &= .96(.15) + P(S|C^2)\ .85 \end{aligned}$$

Hence
$$P(S|C^c) \approx .9835.$$

17. $P(D) = .36$, $P(\text{C}) = .30$, $P(C|D) = .22$

(a) $P(DC) = P(D)\ P(C|D) = .0792$
(b) $P(D|C) = P(DC)/P(C) = .0792/.3 = .264$

18. (a) $$P(\text{Ind}|\text{voted}) = \frac{P(\text{voted}|\text{Ind})P(\text{Ind})}{\sum P(\text{voted}|type)P(\text{type})}$$
$$= \frac{.35(.46)}{.35(.46) + .62(.3) + .58(.24)} \approx 331$$

(b) $$P\{\text{Lib}|\text{voted}\} = \frac{.62(.30)}{.35(.46) + .62(.3) + .58(.24)} \approx .383$$

(c) $$P\{\text{Con}|\text{voted}\} = \frac{.58(.24)}{.35(.46) + .62(.3) + .58(.24)} \approx .286$$

(d) $P\{\text{voted}\} = .35(.46) + .62(.3) + .58(.24) = .4862$
That is, 48.62 percent of the voters voted.

19. Choose a random member of the class. Let A be the event that this person attends the party and let W be the event that this person is a woman.

(a) $P(W|A) = \dfrac{P(A|W)P(W)}{P(A|W)P(W)+P(A|M)P(M)}$ where $M = W^c$

$$= \frac{.48(.38)}{.48(.38)+.37(.62)} \approx .443$$

Therefore, 44.3 percent of the attendees were women.

(b) $P(A) = .48(.38) + .37(.62) = .4118$

Therefore, 41.18 percent of the class attended.

20. (a) $P(F|C) = \dfrac{P(FC)}{P(C)} = .02/.05 = .40$

(b) $P(C|F) = P(FC)/P(F) = .02/.52 = 1/26 \approx .038$

21. (a) $P\{\text{husband under 25}\} = (212 + 36)/500 = .496$

(b) $P\{\text{wife over} | \text{husband over}\} = P\{\text{both over}\}/P\{\text{husband over}\}$

$$= (54/500)/(252/500)$$
$$= 3/14 \approx .214$$

(c) $P\{\text{wife over} | \text{husband under}\} = 36/248 \approx .145$

22. a. $\dfrac{6\cdot 5\cdot 4}{6\cdot 6\cdot 6} = \dfrac{5}{9}$

b. $\dfrac{1}{3!} = \dfrac{1}{6}$

c. $\dfrac{5}{9}\dfrac{1}{6} = \dfrac{5}{54}$

23. $P(w | w \text{ transferred}\}P\{w \text{ tr.}\} + P(w | R \text{ tr.}\}P\{R \text{ tr.}\} = \dfrac{2}{3}\dfrac{1}{3} + \dfrac{1}{3}\dfrac{2}{3} = \dfrac{4}{9}$.

$$P\{w \text{ transferred} | w\} = \frac{P\{w|w \text{ tr.}\}P\{w \text{ tr.}\}}{P\{w\}} = \frac{\frac{2}{3}\frac{1}{3}}{\frac{4}{9}} = 1/2.$$

24. (a) $P\{g-g \mid \text{at least one } g\} = \dfrac{1/4}{3/4} = 1/3$.

(b) Since we have no information about the ball in the urn, the answer is 1/2.

26. Let M be the event that the person is male, and let C be the event that he or she is color blind. Also, let p denote the proportion of the population that is male.

$$P(M \mid C) = \frac{P(C|M)P(M)}{P(C|M)P(M) + P(C|M^c)P(M^c)} = \frac{(.05)p}{(.05)p + (.0025)(1-p)}$$

27. Method (b) is correct as it will enable one to estimate the average number of workers per car. Method (a) gives too much weight to cars carrying a lot of workers. For instance, suppose there are 10 cars, 9 transporting a single worker and the other carrying 9 workers. Then 9 of the 18 workers were in a car carrying 9 workers and so if you randomly choose a worker then with probability 1/2 the worker would have been in a car carrying 9 workers and with probability 1/2 the worker would have been in a car carrying 1 worker.

28. Let A denote the event that the next card is the ace of spades and let B be the event that it is the two of clubs.

(a) $P\{A\} = P\{\text{next card is an ace}\}P\{A \mid \text{next card is an ace}\}$

$$= \frac{3}{32}\frac{1}{4} = \frac{3}{128}$$

(b) Let C be the event that the two of clubs appeared among the first 20 cards.

$$P(B) = P(B \mid C)P(C) + P(B \mid C^c)P(C^c)$$
$$= 0\frac{19}{48} + \frac{1}{32}\frac{29}{48} = \frac{29}{1536}$$

29. Let A be the event that none of the final 3 balls were ever used and let B_i denote the event that i of the first 3 balls chosen had previously been used. Then,

$$P(A) = P(A \mid B_0)P(B_0) + P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + P(A \mid B_3)P(B_3)$$
$$= \sum_{i=0}^{3} \frac{\binom{6+i}{3}}{\binom{15}{3}} \frac{\binom{6}{i}\binom{9}{3-i}}{\binom{15}{3}}$$
$$= .083$$

30. Let B and W be the events that the marble is black and white, respectively, and let B be the event that box i is chosen. Then,

$$P(B) = P(B \mid B_1)P(B_1) + P(B \mid B_2)P(B_2) = (1/2)(1/2) = (2/3)(1/2) = 7/12$$
$$P(B_1 \mid W) = \frac{P(W|B_1)P(B_1)}{P(W)} = \frac{(1/2)(1/2)}{5/12} = 3/5$$

31. Let C be the event that the tumor is cancerous, and let N be the event that the doctor does not call. Then

$$\beta = P(C|N) = \frac{P(NC)}{P(N)}$$
$$= \frac{P(N|C)P(C)}{P(N|C)P(C) + P(N|C^c)P(C^c)}$$
$$= \frac{\alpha}{\alpha + \frac{1}{2}(1-\alpha)}$$
$$= \frac{2\alpha}{1+\alpha} \geq \alpha$$

with strict inequality unless $\alpha = 1$.

32. Let E be the event the child selected is the eldest, and let F_j be the event that the family has j children. Then,

$$P(F_j|E) = \frac{P(EF_j)}{P(E)}$$
$$= \frac{P(F_j)P(E|F_j)}{\sum_j P(F_j)P(E|F_j)}$$
$$= \frac{p_j(1/j)}{.1 + .25(1/2) + .35(1/3) + .3(1/4)} = .24$$

Thus, $P(F_1|E) = .24$, $P(F_4|E) = .18$.

33. Let V be the event that the letter is a vowel. Then

$$P(E|V) = \frac{P(V|E)P(E)}{P(V|E)P(E) + P(V|A)P(A)} = \frac{(1/2)(2/5)}{(1/2)(2/5) + (2/5)(3/5)} = 5/11$$

34. $$P(G|C) = \frac{P(C|G)P(G)}{P(C|G)P(G) + P(C|G^c)P(G^c)} = 54/62$$

35. $P\{A = \text{superior} \mid A \text{ fair}, B \text{ poor}\}$

$$= \frac{P\{A \text{ fair}, B \text{ poor} | A \text{ superior} | A \text{ superior}\}}{P\{A \text{ fair}, B \text{ poor}\}}$$

$$= \frac{\frac{10}{30}\frac{15}{30}\frac{1}{2}}{\frac{10}{30}\frac{15}{30}\frac{1}{2} + \frac{10}{30}\frac{5}{30}\frac{1}{2}} = \frac{3}{4}.$$

36. $P\{C \mid \text{woman}\} = \dfrac{P\{\text{women} \mid C\}P\{C\}}{P\{\text{women} \mid A\}P\{A\} + P\{\text{women} \mid B\}P\{B\} + P\{\text{women} \mid C\}P\{C\}}$

$$= \frac{.7\frac{100}{225}}{.5\frac{50}{225} + .6\frac{75}{225} + .7\frac{100}{225}} = \frac{1}{2}$$

37. (a) $P\{\text{fair} \mid h\} = \dfrac{\frac{1}{2}\frac{1}{2}}{\frac{1}{2}\frac{1}{2} + \frac{1}{2}} = \dfrac{1}{3}$.

(b) $P\{\text{fair} \mid hh\} = \dfrac{\frac{1}{4}\frac{1}{2}}{\frac{1}{4}\frac{1}{2} + \frac{1}{2}} = \dfrac{1}{5}$.

(c) 1

38. $P\{\text{tails} \mid w\} = \dfrac{\frac{3}{15}\frac{1}{2}}{\frac{3}{15}\frac{1}{2} + \frac{5}{12}\frac{1}{2}} = \dfrac{36}{36+75} = \dfrac{36}{111}$.

39. $P\{\text{acc.} \mid \text{no acc.}\} = \dfrac{P\{\text{no acc., acc.}}{P\{\text{no acc.}\}}$

$$= \frac{\frac{3}{10}(.4)(.6) + \frac{7}{10}(.2)(.8)}{\frac{3}{10}(.6) + \frac{7}{10}(.8)} = \frac{46}{185}.$$

40. (a) $\dfrac{7}{12}\dfrac{8}{13}\dfrac{9}{14}$

(b) $3\dfrac{7 \cdot 8 \cdot 5}{12 \cdot 13 \cdot 14}$

(c) $\dfrac{5 \cdot 6 \cdot 7}{12 \cdot 13 \cdot 14}$

(d) $3\dfrac{5 \cdot 6 \cdot 7}{12 \cdot 13 \cdot 14}$

41. $P\{\text{ace}\} = P\{\text{ace} \mid \text{interchanged selected}\} \frac{1}{27}$

$$+P\{\text{ace} \mid \text{interchanged not selected}\} \frac{26}{27}$$

$$= 1\frac{1}{27} + \frac{3}{51}\frac{26}{27} = \frac{129}{51 \cdot 27}.$$

42. $P\{A \mid \text{failure}\} = \dfrac{(.02)(.5)}{(.02)(.5) + (.03)(.3) + (.05)(.2)} = \dfrac{10}{29}$

43. $P\{2 \text{ headed} \mid \text{heads}\} = \dfrac{\frac{1}{3}(1)}{\frac{1}{3}(1) + \frac{1}{3}\frac{1}{2} + \frac{1}{3}\frac{3}{4}} = \dfrac{4}{4+2+3} = \dfrac{4}{9}.$

45. $P\{5\text{th} \mid \text{heads}\} = \dfrac{P\{\text{heads} \mid 5^{\text{th}}\}P\{5^{\text{th}}\}}{\sum_i P\{h \mid i^{\text{th}}\}P\{i^{th}\}}$

$$= \frac{\frac{5}{10}\frac{1}{10}}{\sum_{i=1}^{10} \frac{i}{10}\frac{1}{10}} = \frac{1}{11}.$$

46. Let M and F denote, respectively, the events that the policyholder is male and that the policyholder is female. Conditioning on which is the case gives the following.

$$P(A_2 \mid A_1) = \frac{P(A_1A_2)}{P(A_1)}$$

$$= \frac{P(A_1A_2 \mid M)\alpha + P(A_1A_2 \mid F)(1-\alpha)}{P(A_1 \mid M)\alpha + P(A_1 \mid F)(1-\alpha)}$$

$$= \frac{p_m^2\alpha + p_f^2(1-\alpha)}{p_m\alpha + p_f(1-\alpha)}$$

Hence, we need to show that

$$p_m^2\alpha + p_f^2[1-\alpha) > (p_m\alpha + p_f(1-\alpha))^2$$

or equivalently, that

$$p_m^2(\alpha-\alpha^2) + p_f^2[1-\alpha-(1-a)^2] > 2\alpha(1-\alpha)p_fp_m$$

Factoring out $\alpha(1-\alpha)$ gives the equivalent condition

$$p_m^2 + p_f^2 > 2pf_m$$

or

$$(p_m - p_f)^2 > 0$$

which follows because $p_m \neq p_f$. Intuitively, the inequality follows because given the information that the policyholder had a claim in year 1 makes it more likely that it was a type policyholder having a larger claim probability. That is, the policyholder is more likely to me male if $p_m > p_f$ (or more likely to be female if the inequality is reversed) than without this information, thus raising the probability of a claim in the following year.

47. $$P\{\text{all white}\} = \frac{1}{6}\left[\frac{5}{15} + \frac{5}{15}\frac{4}{14} + \frac{5}{15}\frac{4}{14}\frac{3}{13} + \frac{5}{15}\frac{4}{14}\frac{3}{13}\frac{2}{12} + \frac{5}{15}\frac{4}{14}\frac{3}{13}\frac{2}{12}\frac{1}{11}\right]$$

$$P\{3 \mid \text{all white}\} = \frac{\frac{1}{6}\frac{5}{15}\frac{4}{14}\frac{3}{13}}{P\{\text{all white}\}}$$

48. (a) $P\{\text{silver in other} \mid \text{silver found}\}$

$$= \frac{P\{S \text{ in other}, S \text{ found}\}}{P\{S \text{ found}\}}.$$

To compute these probabilities, condition on the cabinet selected.

$$= \frac{1/2}{P\{S \text{ found} \mid A\}1/2 + P\{S \text{ found} \mid B\}1/2}$$

$$= \frac{1}{1+1/2} = \frac{2}{3}.$$

49. Let C be the event that the patient has cancer, and let E be the event that the test indicates an elevated PSA level. Then, with $p = P(C)$,

$$P(C \mid E) = \frac{P(E \mid C)P(C)}{P(E \mid C)P(C) + P(E \mid C^c)P(C^c)}$$

Similarly,

$$P(C \mid E^c) = \frac{P(E^c \mid C)P(C)}{P(E^c \mid C)P(C) + P(E^c \mid C^c)P(C^c)}$$

$$= \frac{.732p}{.732p + .865(1-p)}$$

50. Choose a person at random

$$P\{\text{they have accident}\} = P\{\text{acc.} \mid \text{good}\}P\{g\} + P\{\text{acc.} \mid \text{ave.}\}P\{\text{ave.}\}$$
$$+ P\{\text{acc.} \mid \text{bad } P(\text{b})\}$$
$$= (.05)(.2) + (.15)(.5) + (.30)(.3) = .175$$

$$P\{A \text{ is good} \mid \text{ no accident}\} = \frac{.95(2)}{.825}$$

$$P\{A \text{ is average} \mid \text{no accident}\} = \frac{(.85)(.5)}{.825}$$

51. Let R be the event that she receives a job offer.

(a) $P(R) = P(R \mid \text{strong})P(\text{strong}) + P(R \mid \text{moderate})P(\text{moderate}) + P(R \mid \text{weak})P(\text{weak})$
$= (.8)(.7) + (.4)(.2) + (.1)(.1) = .65$

(b) $$P(\text{strong} \mid R) = \frac{P(R \mid \text{strong})P(\text{strong})}{P(R)}$$
$$= \frac{(.8)(.7)}{.65} = \frac{56}{65}$$

Similarly,

$$P(\text{moderate} \mid R) = \frac{8}{65},\ P(\text{weak} \mid R) = \frac{1}{65}$$

(c) $$P(\text{strong} \mid R^c) = \frac{P(R^c \mid \text{strong})P(\text{strong})}{P(R^c)}$$
$$= \frac{(.2)(.7)}{.35} = \frac{14}{35}$$

Similarly,

$$P(\text{moderate} \mid R^c) = \frac{12}{35},\ P(\text{weak} \mid R^c) = \frac{9}{35}$$

52. Let M, T, W, Th, F be the events that the mail is received on that day. Also, let A be the event that she is accepted and R that she is rejected.

(a) $P(M) = P(M \mid A)P(A) + P(M \mid R)P(R) = (.15)(.6) + (.05)(.4) = .11$

(b) $P(T \mid M^c) = \dfrac{P(T)}{P(M^c)}$

$$= \frac{P(T|A)P(A) + P(T|R)P(R)}{1 - P(M)}$$

$$= \frac{(.2)(.6) + (.1)(.4)}{.89} \frac{16}{89}$$

(c) $P(A \mid M^c T^c W^c) = \dfrac{P(M^c T^c W^c | A)P(A)}{P(M^c T^c W^c)}$

$$= \frac{(1 - .15 - .20 - .25)(.6)}{(.4)(.6) + (.75)(.4)} = \frac{12}{27}$$

(d) $P(A \mid Th) = \dfrac{P(Th|A)P(A)}{P(Th)}$

$$= \frac{(.15)(.6)}{(.15)(.6) + (.15)(.4)} = \frac{3}{5}$$

(e) $P(A \mid \text{no mail}) = \dfrac{P(\text{no mail}|A)P(A)}{P(\text{no mail})}$

$$= \frac{(.15)(.6)}{(.15)(.6) + (.4)(.4)} = \frac{9}{25}$$

53. Let W and F be the events that component 1 works and that the system functions.

$$P(W \mid F) = \frac{P(WF)}{P(F)} = \frac{P(W)}{1 - P(F^c)} = \frac{1/2}{1 - (1/2)^{n-1}}$$

55. $P\{\text{Boy}, F\} = \dfrac{4}{16 + x}$ $\quad P\{\text{Boy}\} = \dfrac{10}{16 + x}$ $\quad P\{F\} = \dfrac{10}{16 + x}$

so independence $\Rightarrow 4 = \dfrac{10 \cdot 10}{16 + x} \Rightarrow 4x = 36$ or $x = 9$.

A direct check now shows that 9 sophomore girls (which the above shows is necessary) is also sufficient for independence of sex and class.

56. $P\{\text{new}\} = \sum_i P\{\text{new} \mid \text{type } i\} p_i = \sum_i (1 - p_i)^{n-1} p_i$

57. (a) $2p(1 - p)$

(b) $\binom{3}{2}p^2(1-p)$

(c) $P\{\text{up on first} \mid \text{up 1 after 3}\}$
$= P\{\text{up first, up 1 after 3}\}/[3p^2(1 - p)]$
$= p2p(1 - p)/[3p^2(1 - p)] = 2/3.$

58. (a) All we know when the procedure ends is that the two most flips were either *H*, *T*, or *T*, *H*. Thus,

$$P(\text{heads}) = P(H, T \mid H, T \text{ or } T, H)$$
$$= \frac{P(H,T)}{P(H,T)+P(T,H)} = \frac{p(1-p)}{p(1-p)+(1-p)p} = \frac{1}{2}$$

(b) No, with this new procedure the result will be heads (tails) whenever the first flip is tails (heads). Hence, it will be heads with probability $1 - p$.

59. (a) 1/16

(b) 1/16

(c) The only way in which the pattern *H*, *H*, *H*, *H* can occur first is for the first 4 flips to all be heads, for once a tail appears it follows that a tail will precede the first run of 4 heads (and so *T*, *H*, *H*, *H* will appear first). Hence, the probability that *T*, *H*, *H*, *H* occurs first is 15/16.

60. From the information of the problem we can conclude that both of Smith's parents have one blue and one brown eyed gene. Note that at birth, Smith was equally likely to receive either a blue gene or a brown gene from each parent. Let *X* denote the number of blue genes that Smith received.

(a) $P\{\text{Smith blue gene}\} = P\{X = 1 \mid X \le 1\} = \frac{1/2}{1-1/4} = 2/3$

(b) Condition on whether Smith has a blue-eyed gene.
$P\{\text{child blue}\} = P\{\text{blue} \mid \text{blue gene}\}(2/3) + P\{\text{blue} \mid \text{no blue}\}(1/3)$
$= (1/2)(2/3) = 1/3$

(c) First compute

$$P\{\text{Smith blue} \mid \text{child brown}\} = \frac{P\{\text{child brown} \mid \text{Smith blue}\}2/3}{2/3}$$
$$= 1/2$$

Now condition on whether Smith has a blue gene given that first child has brown eyes.
$P\{\text{second child brown}\} = P\{\text{brown} \mid \text{Smith blue}\}1/2 + P\{\text{brown} \mid \text{Smith no blue}\}1/2$
$= 1/4 + 1/2 = 3/4$

61. Because the non-albino child has an albino sibling we know that both its parents are carriers. Hence, the probability that the non-albino child is not a carrier is

$$P(A, A \mid A, a \text{ or } a, A \text{ or } A, A) = \frac{1}{3}$$

Where the first gene member in each gene pair is from the mother and the second from the father. Hence, with probability 2/3 the non-albino child is a carrier.

(a) Condition on whether the non-albino child is a carrier. With C denoting this event, and O_i the event that the i^{th} offspring is albino, we have:

$$\begin{aligned} P(O_1) &= P(O_1 \mid C)P(C) + P(O_1 \mid C^c)P(C^c) \\ &= (1/4)(2/3) + 0(1/3) = 1/6 \end{aligned}$$

(b)

$$\begin{aligned} P(O_2 \mid O_1^c) &= \frac{P(O_1^c O_2)}{P(O_1^c)} \\ &= \frac{P(O_1^c O_2 \mid C)P(C) + P(O_1^c O_2 \mid C^c)P(C^c)}{5/6} \\ &= \frac{(3/4)(1/4)(2/3) + 0(1/3)}{5/6} = \frac{3}{20} \end{aligned}$$

62. (a)

$$\begin{aligned} P\{\text{both hit} \mid \text{at least one hit}\} &= \frac{P\{\text{both hit}\}}{P\{\text{at least one hit}\}} \\ &= p_1 p_2/(1 - q_1 q_2) \end{aligned}$$

(b) $P\{\text{Barb hit} \mid \text{at least one hit}\} = p_1/(1 - q_1 q_2)$

$Q_i = 1 - p_i$, and we have assumed that the outcomes of the shots are independent.

63. Consider the final round of the duel. Let $q_x = 1 - p_x$

(a)

$$\begin{aligned} P\{A \text{ not hit}\} &= P\{A \text{ not hit} \mid \text{at least one is hit}\} \\ &= P\{A \text{ not hit}, B \text{ hit}\}/P\{\text{at least one is hit}\} \\ &= q_B p_A/(1 - q_A q_B) \end{aligned}$$

(b)

$$\begin{aligned} P\{\text{both hit}\} &= P\{\text{both hit} \mid \text{at least one is hit}\} \\ &= P\{\text{both hit}\}/P\{\text{at least one hit}\} \\ &= p_A p_B/(1 - q_A q_B) \end{aligned}$$

(c) $(q_A q_B)^{n-1}(1 - q_A q_B)$

(d)

$$\begin{aligned} P\{n \text{ rounds} \mid A \text{ unhit}\} &= P\{n \text{ rounds}, A \text{ unhit}\}/P\{A \text{ unhit}\} \\ &= \frac{(q_A q_B)^{n-1} p_A q_B}{q_B p_A/(1 - q_A q_B)} \\ &= (q_A q_B)^{n-1}(1 - q_A q_B) \end{aligned}$$

(e) $P(n \text{ rounds} \mid \text{both hit}\} = P\{n \text{ rounds both hit}\}/P\{\text{both hit}\}$

$$= \frac{(q_A q_B)^{n-1} p_A p_B}{p_B p_A /(1 - q_A q_B)}$$

$$= (q_A q_B)^{n-1}(1 - q_A q_B)$$

Note that (c), (d), and (e) all have the same answer.

64. If use (a) will win with probability p. If use strategy (b) then

$$P\{\text{win}\} = P\{\text{win} \mid \text{both correct}\}p^2 + P\{\text{win} \mid \text{exactly 1 correct}\}2p(1-p) + P\{\text{win} \mid \text{neither correct}\}(1-p)^2$$
$$= p^2 + p(1-p) + 0 = p$$

Thus, both strategies give the same probability of winning.

65. (a) $P\{\text{correct} \mid \text{agree}\} = P\{\text{correct, agree}\}/P\{\text{agree}\}$
$= p^2/[p^2 + (1-p)^2]$
$= 36/52 = 9/13$ when $p = .6$

(b) 1/2

66. (a) $[I - (1 - P_1P_2)(1 - P_3P_4)]P_5 = (P_1P_2 + P_3P_4 - P_1P_2P_3P_4)P_5$

(b) Let $E_1 = \{1 \text{ and } 4 \text{ close}\}$, $E_2 = \{1, 3, 5 \text{ all close}\}$

$E_3 = \{2, 5 \text{ close}\}$, $E_4 = \{2, 3, 4 \text{ close}\}$. The desired probability is

67. $$P(E_1 \cup E_2 \cup E_3 \cup E_4) = P(E_1) + P(E_2) + P(E_3) + P(E_4) - P(E_1E_2) - P(E_1E_3) - P(E_1E_4) - P(E_2E_3) - P(E_2E_4) + P(E_3E_4) + P(E_1E_2E_3) + P(E_1E_2E_4) + P(E_1E_3E_4) + P(E_2E_3E_4) - P(E_1E_2E_3E_4)$$

$$= P_1P_4 + P_1P_3P_5 + P_2P_5 + P_2P_3P_4 - P_1P_3P_4P_5 - P_1P_2P_4P_5 - P_1P_2P_3P_4 - P_1P_2P_3P_5 - P_2P_3P_4P_5 - 2P_1P_2P_3P_4P_5 + 3P_1P_2P_3P_4P_5.$$

(a) $P_1P_2(1 - P_3)(1 - P_4) + P_1(1 - P_2)P_3(1 - P_4) + P_1(1 - P_2(1 - P_3)P_4$
$+ P_2P_3(1 - P_1)(1 - P_4) + (1 - P_1)P_2(1 - P_3)P_4 + (1 - P_1)(1 - P_2)P_3P_4$
$+ P_1P_2P_3(1 - P_4) + P_1P_2(1 - P_3)P_4 + P_1(1 - P_2)P_3P_4 + (1 - P_1)P_2P_3P_4 + P_1P_2P_3P_4.$

(c) $$\sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

68. Let C_i denote the event that relay i is closed, and let F be the event that current flows from A to B.

$$P(C_1C_2 \mid F) = \frac{P(C_1C_2F)}{P(F)}$$

$$= \frac{P(F \mid C_1C_2)P(C_1C_2)}{p_5(p_1p_2 + p_3p_4 - p_1p_2p_3p_4)}$$

$$= \frac{p_5p_1p_2}{p_5(p_1p_2 + p_3p_4 - p_1p_2p_3p_4)}$$

69. 1. (a) $\frac{1}{2}\frac{3}{4}\frac{1}{2}\frac{3}{4}\frac{1}{2} = \frac{9}{128}$

(b) $\frac{1}{2}\frac{3}{4}\frac{1}{2}\frac{3}{4}\frac{1}{2} = \frac{9}{128}$

(c) $\frac{18}{128}$

(d) $\frac{110}{128}$

2. (a) $\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{32}$

(b) $\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{32}$

(c) $\frac{1}{16}$

(d) $\frac{15}{16}$

70. (a) $P\{\text{carrier} \mid 3 \text{ without}\}$

$$= \frac{1/8\,1/2}{1/8\,1/2 + 1\,1/2} = 1/9.$$

(b) 1/18

71. $P\{\text{Braves win}\} = P\{B \mid B \text{ wins 3 of 3}\}\ 1/8 + P\{B \mid B \text{ wins 2 of 3}\}\ 3/8$
$+ P\{B \mid B \text{ wins 1 of 3}\}\ 3/8 + P\{B \mid B \text{ wins 0 of 3}\}\ 1/8$

$$= \frac{1}{8} + \frac{3}{8}\left[\frac{1}{4}\frac{1}{2} + \frac{3}{4}\right] + \frac{3}{8}\quad\frac{3}{4}\frac{1}{2} = \frac{38}{64}$$

where $P\{B \mid B \text{ wins } i \text{ of 3}\}$ is obtained by conditioning on the outcome of the other series. For instance

$$P\{B \mid B \text{ win 2 of 3}\} = P\{B \mid D \text{ or } G \text{ win 3 of 3}, B \text{ win 2 of 3}\}\ 1/4$$
$$= P\{B \mid D \text{ or } G \text{ win 2 of 3}, B \text{ win 2 of 3}\}\ 3/4$$
$$= \frac{1}{2}\frac{1}{4} + \frac{3}{4}.$$

By symmetry $P\{D \text{ win}\} = P\{G \text{ win}\}$ and as the probabilities must sum to 1 we have.

$$P\{D \text{ win}\} = P\{G \text{ win}\} = \frac{13}{64}.$$

72. Let f denote for and a against a certain place of legislature. The situations in which a given steering committees vote is decisive are as follows:

given member	other members of S.C.	other council members
for	both for	3 or 4 against
for	one for, one against	at least 2 for
against	one for, one against	at least 2 for
against	both for	3 of 4 against

$$P\{\text{decisive}\} = p^3 4p(1-p)^3 + p^2 p(1-p)(6p^2(1-p)^2 + 4p^3(1-p) + p^4) + (1-p)2p(1-p)(6p^2(1-p)^2 + 4p^3(1-p) + p^4) + (1-p)p^2 4p(1-p)^3.$$

73. (a) 1/16, (b) 1/32, (c) 10/32, (d) 1/4, (e) 31/32.

74. Let P_A be the probability that A wins when A rolls first, and let P_B be the probability that B wins when B rolls first. Using that the sum of the dice is 9 with probability 1/9, we obtain upon conditioning on whether A rolls a 9 that

$$P_A = \frac{1}{9} + \frac{8}{9}(1 - P_B)$$

Similarly,

$$P_B = \frac{5}{36} + \frac{31}{36}(1 - P_A)$$

Solving these equations gives that $P_A = 9/19$ (and that $P_B = 45/76$.)

75. (a) The probability that a family has 2 sons is 1/4; the probability that a family has exactly 1 son is 1/2. Therefore, on average, every four families will have one family with 2 sons and two families with 1 son. Therefore, three out of every four sons will be eldest sons.

Another argument is to choose a child at random. Letting E be the event that the child is an eldest son, letting S be the event that it is a son, and letting A be the event that the child's family has at least one son,

$$\begin{aligned} P(E \mid S) &= \frac{P(ES)}{P(S)} \\ &= 2P(E) \\ &= 2\left[P(E|A)\frac{3}{4} + P(E|A^c)\frac{1}{4}\right] \\ &= 2\left[\frac{1}{2}\frac{3}{4} + 0\frac{1}{4}\right] = 3/4 \end{aligned}$$

(b) Using the preceding notation

$$\begin{aligned}P(E \mid S) &= \frac{P(ES)}{P(S)} \\ &= 2P(E) \\ &= 2\left[P(E|A)\frac{7}{8} + P(E|A^c)\frac{1}{8}\right] \\ &= 2\left[\frac{1}{3}\frac{7}{8}\right] = 7/12\end{aligned}$$

76. Condition on outcome of initial trial

$$\begin{aligned}P(E \text{ before } F) &= P(E \text{ b } F \mid E)P(E) + P(E \text{ b } F \mid F)P(F) \\ &\quad + P(E \text{ b } F \mid \text{neither } E \text{ or } F)[1 - P(E) - P(F)] \\ &= P(E) + P(E \text{ b } F)(1 - P(E) - P(F)].\end{aligned}$$

Hence,

$$P(E \text{ b } F) = \frac{P(E)}{P(E)+P(F)}.$$

77. (a) This is equal to the conditional probability that the first trial results in outcome 1 (F_1) given that it results in either 1 or 2, giving the result 1/2. More formally, with L_3 being the event that outcome 3 is the last to occur

$$P(F_1 \mid L_3) = \frac{P(L_3|F_1)P(F_1)}{P(L_3)} = \frac{(1/2)(1/3)}{1/3} = 1/2$$

(b) With S_1 being the event that the second trial results in outcome 1, we have

$$P(F_1S_1 \mid L_3) = \frac{P(L_3|F_1S_1)P(F_1S_1)}{P(L_3)} = \frac{(1/2)(1/9)}{1/3} = 1/6$$

78. (a) Because there will be 4 games if each player wins one of the first two games and then one of them wins the next two, $P(4 \text{ games}) = 2p(1-p)[p^2 + (1-p)^2]$.

(b) Let A be the event that A wins. Conditioning on the outcome of the first two games gives

$$\begin{aligned}P(A &= P(A \mid a, a)p^2 + P(A \mid a, b)p(1-p) + P(A \mid b, a)(1-p)p + P(A \mid b, b)(1-p)^2 \\ &= p^2 + P(A)2p(1-p)\end{aligned}$$

where the notation a, b means, for instance, that A wins the first and B wins the second game. The final equation used that $P(A \mid a, b) = P(A \mid b, a) = P(A)$. Solving, gives

$$P(A) = \frac{p^2}{1-2p(1-p)}$$

79. Each roll that is either a 7 or an even number will be a 7 with probability

$$p = \frac{P(7)}{P(7)+P(\text{even})} = \frac{1/6}{1/6+1/2} = 1/4$$

Hence, from Example 4*i* we see that the desired probability is

$$\sum_{i=2}^{7}\binom{7}{i}(1/4)^i(3/4)^{7-i} = 1 - (3/4)^7 - 7(3/4)^6(1/4)$$

80. (a) $P(A_i) = (1/2)^i$, if $i < n$
$= (1/2)^{n-1}$, if $i = n$

(b) $$\frac{\sum_{i=1}^{n} i(1/2)^i + n(1/2)^{n-1}}{2^n - 1} = \frac{1}{2^{n-1}}$$

(c) Condition on whether they initially play each other. This gives

$$P_n = \frac{1}{2^n-1} + \frac{2^n-2}{2^n-1}\left(\frac{1}{2}\right)^2 P_{n-1}$$

where $\left(\frac{1}{2}\right)^2$ is the probability they both win given they do not play each other.

(d) There will be $2^n - 1$ losers, and thus that number of games.

(e) Since the 2 players in game i are equally likely to be any of the $\binom{2^n}{2}$ pairs it follows that

$P(B_i) = 1\Big/\binom{2^n}{2}$.

(f) Since the events B_i are mutually exclusive

$$P(\cup B_i) = \sum P(B_i) = (2^n - 1)\Big/\binom{2^n}{2} = (1/2)^{n-1}$$

81. $$\frac{1-(9/11)^{15}}{1-(9/11)^{30}}$$

82. (a) $P(A) = P_1^2 + (1-P_1^2)[(1-P_2^2)P(A)]$ or $P(A) = \dfrac{P_1^2}{P_1^2 + P_2^2 - P_1^2P_2^2}$

(c) similar to (a) with P_i^3 replacing P_i^2.

(b) and (d) Let $P_{ij}(\overline{P}_{ij})$ denote the probability that A wins when A needs i more and B needs j more and $A(B)$ is to flip. Then

$$P_{ij} = P_1 P_{i-1,j} + (1-P_1)\overline{P}_{ij}$$
$$\overline{P}_{ij} = P_2 \overline{P}_{i,j-1} + (1-P_2)P_{ij}.$$

These equations can be recursively solved starting with

$$P_{01} = 1,\ P_{1,0} = 0.$$

83. (a) Condition on the coin flip

$$P\{\text{throw } n \text{ is red}\} = \frac{1}{2}\frac{4}{6} + \frac{1}{2}\frac{2}{6} = \frac{1}{2}$$

(b) $$P\{r \mid rr\} = \frac{P\{rrr\}}{P\{rr\}} = \frac{\frac{1}{2}\left(\frac{2}{3}\right)^3 + \frac{1}{2}\left(\frac{1}{3}\right)^3}{\frac{1}{2}\left(\frac{2}{3}\right)^2 + \frac{1}{2}\left(\frac{1}{3}\right)^2} = \frac{3}{5}$$

(c) $$P\{A \mid rr\} = \frac{P\{rr \mid A\}P(A)}{P\{rr\}} = \frac{\left(\frac{2}{3}\right)^2 \frac{1}{2}}{\left(\frac{2}{3}\right)^2\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)^2 \frac{1}{2}} = 4/5$$

84. (b) $$P(A \text{ wins}) = \frac{4}{12} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{4}{9} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{4}{6} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}$$
$$P(B \text{ wins}) = \frac{8}{12}\frac{4}{11} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{4}{5}$$
$$P(C \text{ wins}) = \frac{8}{12}\frac{7}{11}\frac{4}{10} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{4}{7} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{1}{5}$$

85. Part (a) remains the same. The possibilities for part (b) become more numerous.

86. Using the hint

$$P\{A \subset B\} = \sum_{i=0}^{n} (2^i/2^n)\binom{n}{i}\Big/2^n = \sum_{i=0}^{n}\binom{n}{i}2^i/4^n = (3/4)^n$$

where the final equality uses

$$\sum_{i=0}^{n}\binom{n}{i}2^i 1^{n-i} = (2+1)^n$$

(b) $P(AB = \phi) = P(A \subset B^c) = (3/4)^n$, by part (a), since B^c is also equally likely to be any of the subsets.

87. $P\{i^{\text{th}} \mid \text{all heads}\} = \dfrac{(i/k)^n}{\sum_{j=0}^{k}(j/k)^n}$.

88. No—they are conditionally independent given the coin selected.

89. (a) $P(J_3$ votes guilty $\mid J_1$ and J_2 vote guilty$\}$

$= P\{J_1, J_2, J_3$ all vote guilty$\}/P\{J_1$ and J_2 vote guilty$\}$

$$= \frac{\frac{7}{10}(.7)^3 + \frac{3}{10}(.2)^3}{\frac{7}{10}(.7)^2 + \frac{3}{10}(.2)^2} = \frac{97}{142}.$$

(b) $P(J_3$ guilty $\mid$ one of J_1, J_2 votes guilty$\}$

$$= \frac{\frac{7}{10}(.7)2(.7)(.3) + \frac{3}{10}(2.)2(.2)(.8)}{\frac{7}{10}2(.7)(.3) + \frac{3}{10}2(.2)(.8)} = \frac{15}{26}.$$

(c) $P\{J_3$ guilty $\mid J_1, J_2$ vote innocent$\}$

$$= \frac{\frac{7}{10}(.7)(.3)^2 + \frac{3}{10}(.2)(.8)^2}{\frac{7}{10}(.3)^2 + \frac{3}{10}(.8)^2} = \frac{33}{102}.$$

E_i are conditionally independent given the guilt or innocence of the defendant.

90. Let N_i denote the event that none of the trials result in outcome i, $i = 1, 2$. Then

$$\begin{aligned} P(N_1 \cup N_2) &= P(N_1) + P(N_2) - P(N_1N_2) \\ &= (1 - p_1)^n + (1 - p_2)^n - (1 - p_1 - p_2)^n \end{aligned}$$

Hence, the probability that both outcomes occur at least once is
$1 - (1 - p_1)^n - (1 - p_2)^n + (p_0)^n$.

Theoretical Exercises

1. $$P(AB \mid A) = \frac{P(AB)}{P(A)} \geq \frac{P(AB)}{P(A \cup B)} = P(AB \mid A \cup B)$$

2. If $A \subset B$

$$P(A \mid B) = \frac{P(A)}{P(B)}, \; P(A \mid B^c) = 0, \qquad P(B \mid A) = 1, \qquad P(B \mid A^c) = \frac{P(BA^c)}{P(A^c)}$$

3. Let F be the event that a first born is chosen. Also, let S_i be the event that the family chosen in method a is of size i.

$$P_a(F) = \sum_i P(F \mid S_i)P(S_i) = \sum_i \frac{1}{i}\frac{n_i}{m}$$

$$P_b(F) = \frac{m}{\sum_i in_i}$$

Thus, we must show that

$$\sum_i in_i \sum_i n_i / i \geq m^2$$

or, equivalently,

$$\sum_i in_i \sum_j n_j / j \geq \sum_i n_i \sum_j n_j$$

or,

$$\sum\sum_{i \neq j} \frac{i}{j} n_i n_j \geq \sum\sum_{i \neq j} n_i n_j$$

Considering the coefficients of the term $n_i n_j$, shows that it is sufficient to establish that

$$\frac{i}{j} + \frac{j}{i} \geq 2$$

or equivalently

$$i^2 + j^2 \geq 2ij$$

which follows since $(i - j)^2 \geq 0$.

4. Let N_i denote the event that the ball is not found in a search of box i, and let B_j denote the event that it is in box j.

$$P(B_j \mid N_i) = \frac{P(N_i \mid B_j)P(B_j)}{P(N_i \mid B_i)P(B_i) + P(N_i \mid B_i^c)P(B_i^c)}$$

$$= \frac{P_j}{(1-\alpha_i)P_i + 1 - P_i} \quad \text{if } j \neq i$$

$$= \frac{(1-\alpha_i)P_i}{(1-\alpha_i)P_i + 1 - P_i} \quad \text{if } j = i$$

5. None are true.

6. $$P\left(\bigcup_1^n E_i\right) = 1 - P\left(\bigcap_1^n E_i^c\right) = 1 - \prod_1^n [1 - P(E_i)]$$

7. (a) They will all be white if the last ball withdrawn from the urn (when all balls are withdrawn) is white. As it is equally likely to by any of the $n + m$ balls the result follows.

(b) $P(RBG) = \dfrac{g}{r+b+g} P(RBG \mid G \text{ last}) = \dfrac{g}{r+b+g}\dfrac{b}{r+b}$.

Hence, the answer is $\dfrac{bg}{(r+b)(r+b+g)} + \dfrac{b}{r+b+g}\dfrac{g}{r+g}$.

8. (a) $P(A) = P(A \mid C)P(C) + P(A \mid C^c)P(C^c) > P(B \mid C)P(C) + P(B \mid C^c)P(C^c) = P(B)$

(b) For the events given in the hint

$$P(A \mid C) = \frac{P(C \mid A)P(A)}{3/36} = \frac{(1/6)(1/6)}{3/36} = 1/3$$

Because $1/6 = P(A$ is a weighted average of $P(A \mid C)$ and $P(A \mid C^c)$, it follows from the result $P(A \mid C) > P(A)$ that $P(A \mid C^c) < P(A)$. Similarly,

$$1/3 = P(B \mid C) > P(B) > P(B \mid C^c)$$

However, $P(AB \mid C) = 0 < P(AB \mid C^c)$.

9. $P(A) = P(B) = P(C) = 1/2$, $P(AB) = P(AC) = P(BC) = 1/4$. But, $P(ABC) = 1/4$.

10. $P(A_{i,j}) = 1/365$. For $i \neq j \neq k$, $P(A_{i,j}A_{j,k}) = 365/(365)^3 = 1/(365)^2$. Also, for $i \neq j \neq k \neq r$, $P(A_{i,j}A_{k,r}) = 1/(365)^2$.

11. $1 - (1-p)^n \geq 1/2$, or, $n \geq -\dfrac{\log(2)}{\log(1-p)}$

12. $a_i \prod_{j=1}^{i-1}(1-a_j)$ is the probability that the first head appears on the i^{th} flip and $\prod_{i=1}^{\infty}(1-a_i)$ is the probability that all flips land on tails.

13. Condition on the initial flip. If it lands on heads then A will win with probability $P_{n-1,m}$ whereas if it lands tails then B will win with probability $P_{m,n}$ (and so A will win with probability $1 - P_{m,n}$).

14. Let N go to infinity in Example 4j.

15. $P\{r$ successes before m failures$\}$
$= P\{r^{\text{th}}$ success occurs before trial $m + r\}$
$$= \sum_{n=r}^{m+r-1} \binom{n-1}{r-1} p^r (1-p)^{n-r} .$$

16. If the first trial is a success, then the remaining $n - 1$ must result in an odd number of successes, whereas if it is a failure, then the remaining $n - 1$ must result in an even number of successes.

17. $P_1 = 1/3$
$P_2 = (1/3)(4/5) + (2/3)(1/5) = 2/5$
$P_3 = (1/3)(4/5)(6/7) + (2/3)(4/5)(1/7) + (1/3)(1/5)(1/7) = 3/7$
$P_4 = 4/9$

(b) $P_n = \dfrac{n}{2n+1}$

(c) Condition on the result of trial n to obtain

$$P_n = (1 - P_{n-1})\frac{1}{2n+1} + P_{n-1}\frac{2n}{2n+1}$$

(d) Must show that

$$\frac{n}{2n+1} = \left[1 - \frac{n-1}{2n-1}\right]\frac{1}{2n+1} + \frac{n-1}{2n-1}\frac{2n}{2n+1}$$

or equivalently, that

$$\frac{n}{2n+1} = \frac{n}{2n-1}\frac{1}{2n+1} + \frac{n-1}{2n-1}\frac{2n}{2n+1}$$

But the right hand side is equal to

$$\frac{n+2n(n-1)}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

18. Condition on when the first tail occurs.

19. $P_{n,i} = p^{P}_{n-1,i+1} + (1-p)P_{n-1,i-1}$

20. $\alpha_{n+1} = \alpha_n p + (1 - \alpha_n)(1 - p^1)$
$P_n = \alpha_n p + (1 - \alpha_n)p^1$

21. (b) $P_{n,1} = P\{A \text{ receives first 2 votes}\} = \dfrac{n(n-1)}{(n+1)n} = \dfrac{n-1}{n+1}$

$P_{n,2} = P\{A \text{ receives first 2 and at least 1 of the next 2}\}$

$$= \frac{n}{n+2}\frac{n-1}{n+1}\left\{1 - \frac{2\cdot 1}{n(n-1)}\right\} = \frac{n-2}{n+2}$$

(c) $P_{n,m} = \dfrac{n-m}{n+m}$, $n \geq m$.

(d) $P_{n,m} = P\{A \text{ always ahead}\}$

$$= P\{A \text{ always} \mid A \text{ receives last vote}\}\ \frac{n}{n+m}$$

$$+ P\{A \text{ always} \mid B \text{ receives last vote}\}\ \frac{m}{n+m}$$

$$= \frac{n}{n+m}P_{n-1,m} + \frac{m}{n+m}P_{n,m-1}$$

(e) The conjecture of (c) is true when $n + m = 1$ ($n = 1$, $m = 0$).
Assume it when $n + m = k$. Now suppose that $n + m = k + 1$. By (d) and the induction hypothesis we have that

$$P_{n,m} = \frac{n}{n+m}\frac{n-1-m}{n-1+m} + \frac{m}{n+m}\frac{n-m+1}{n+m-1} = \frac{n-m}{n+m}$$

which completes the proof.

22. $P_n = P_{n-1}p + (1 - P_{n-1})(1 - p)$

$= (2p - 1)P_{n-1} + (1 - p)$

$= (2p-1)\left[\dfrac{1}{2} + \dfrac{1}{2}(2p-1)^{n-1}\right] + 1 - p$ by the induction hypothesis

$= \dfrac{2p-1}{2} + \dfrac{1}{2}(2p-1)^n + 1 - p$

$= \dfrac{1}{2} + \dfrac{1}{2}(2p-1)^n$.

23. $P_{1,1} = 1/2$. Assume that $P_{a,b} = 1/2$ when $k \ge a + b$ and now suppose $a + b = k + 1$. Now

$$P_{a,b} = P\{\text{last is white} \mid \text{first } a \text{ are white}\} \frac{1}{\binom{a+b}{a}}$$

$$+ P\{\text{last is white} \mid \text{first } b \text{ are black}\} \frac{1}{\binom{b+a}{b}}$$

$$+ P\{\text{last is white} \mid \text{neither first } a \text{ are white nor first } b \text{ are black}\}$$

$$\left[1 - \frac{1}{\binom{a+b}{a}} - \frac{1}{\binom{b+a}{b}}\right] = \frac{a!b!}{(a+b)!} + \frac{1}{2}\left[1 - \frac{a!b!}{(a+b)!} - \frac{a!b!}{(a+b)!}\right] = \frac{1}{2}$$

where the induction hypothesis was used to obtain the final conditional probability above.

24. The probability that a given contestant does not beat all the members of some given subset of k other contestants is, by independence, $1 - (1/2)^k$. Therefore $P(B_i)$, the probability that none of the other $n - k$ contestants beats all the members of a given subset of k contestants, is $[1 - (1/2)^k]^{n-k}$. Hence, Boole's inequality we have that

$$P(\cup B_i) \le \binom{n}{k}[1 - (1/2)^k]^{n-k}$$

Hence, if $\binom{n}{k}[1 - (1/2)^k]^{n-k} < 1$ then there is a positive probability that none of the $\binom{n}{k}$ events B_i occur, which means that there is a positive probability that for every set of k contestants there is a contestant who beats each member of this set.

25. $P(E \mid F) = P(EF)/P(F)$

$$P(E \mid FG)P(G \mid F) = \frac{P(EFG)}{P(FG)}\frac{P(FG)}{P(F)} = \frac{P(EFG)}{P(F)}$$

$$P(E \mid FG^c)P(G^c \mid F) = \frac{P(EFG^c)}{P(F)}.$$

The result now follows since

$$P(EF) = P(EFG) + P(EFG^c)$$

27. $E_1, E_2, \ldots, E_n$ are conditionally independent given F if for all subsets $i_1, \ldots, i_r$ of $1, 2, \ldots, n$

$$P\left(E_{i_1} \ldots E_{i_r} \middle| F\right) = \prod_{j=1}^{r} P\left(E_{i_j|F}\right).$$

28. Not true. Let $F = E_1$.

29. $P\{\text{next } m \text{ heads} \mid \text{first } n \text{ heads}\}$
$= P\{\text{first } n + m \text{ are heads}\}/P(\text{first } n \text{ heads}\}$
$$= \int_0^1 p^{n+m}dp \Big/ \int_0^1 p^n dp = \frac{n+1}{n+m+1}.$$