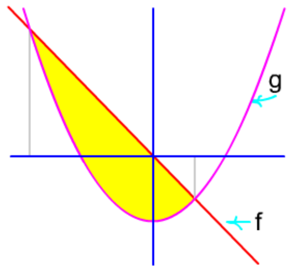
CHAPTER 6: Applications of integration

6.1 Areas Between Curves

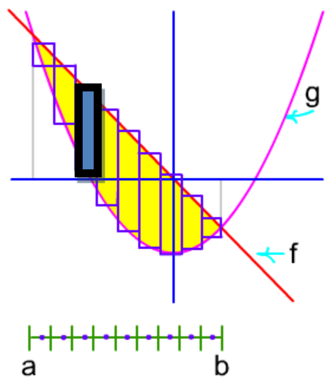
Suppose we want to compute the area bounded by two curves, $y = f(x)$ and $y = g(x)$, with $f(x) \geq g(x)$:



Note: We can find the points of intersection by solving $f(x) = g(x)$. This gives us the interval [a,b].

We'll develop the formula for the area by applying the very useful method of Riemann Sums:

1) Subdivide [a,b] into n subintervals of length Δx
2) For each $i = 1,2, \dots n$ pick a point x_i in the i^{th} subinterval, and draw a line segment from $(x_i, f(x_i))$ to $(x_i, g(x_i))$.
3) Draw rectangles using these segments as heights, and Δx as widths.



Note that the height of the i^{th} rectangle is $f(x_i) - g(x_i)$, so the area of the i^{th} rectangle is $(f(x_i) - g(x_i))\Delta x$

Therefore, the sum of the areas of all the rectangles (which is an approximation of the area we want) is:

$$\sum_{i=1}^{n} (f(x_i) - g(x_i))\Delta x$$

The larger the number of approximating rectangles, the better the approximation.

4) We DEFINE the area bounded by the graphs of f and g to be:

$$Area = \lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i) - g(x_i))\Delta x$$

If f and g are continuous on [a,b], this limit corresponds to the definite integral:

$$\boxed{Area = \int_a^b f(x) - g(x)\, dx}$$

Basic Example: ***Find the area enclosed by the curves $y = 12 - x^2$ and $y = x^2 - 6$.***

a. The two curves cross at:

b. To determine which curve is above, you can sketch their graphs

OR: Since the functions are continuous, they cannot switch sides without crossing. So, we can check which is greater at a point between the crossing point (for ex: $f(0) = 12,\ g(0) = -6$, so $f(x) = 12 - x^2$ is above)

c. $\text{Area bounded by the curves } = \int_{-3}^{3} f(x) - g(x)\ dx = 72$

Work:

Important:

1) The general formula (regardless which function is above) is:

$$\boldsymbol{Area = \int_a^b |f(x) - g(x)|\, dx}$$

So, if the functions on the top and bottom change during the interval, we need to split the interval into subintervals according to the formula changes, compute each integral separately, then add.

For example: Compute the area between $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi$

a. Crossing points: on the interval $[0, \pi]$, $\sin x = \cos x$ at $x =$ ______

b. From 0 to $\frac{\pi}{4}$, the function __________ is higher.
From $\frac{\pi}{4}$ to π, the function __________ is higher.
(sketch the graphs or evaluate at, say, $x = 0, x = \frac{\pi}{2}$)

c. So: $Area = \int_0^{\pi} |\sin x - \cos x|\, dx = \int_0^{\frac{\pi}{4}} (\qquad\qquad)\, dx + \int_{\frac{\pi}{4}}^{\pi} (\qquad\qquad)\, dx =$

$$= \quad (\sin x + \cos x)|_0^{\pi/4} \quad + \quad (-\cos x - \sin x)\,|_{\pi/4}^{\pi}$$

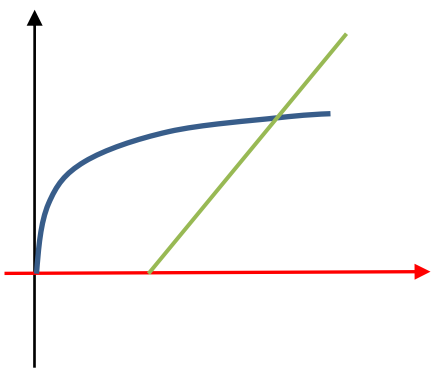
$$= \left[\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1)\right] \; + \; \left[(-(-1) - 0) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right]$$

$$= [\sqrt{2} - 1] + [1 + \sqrt{2}] = \boxed{2\sqrt{2}}$$

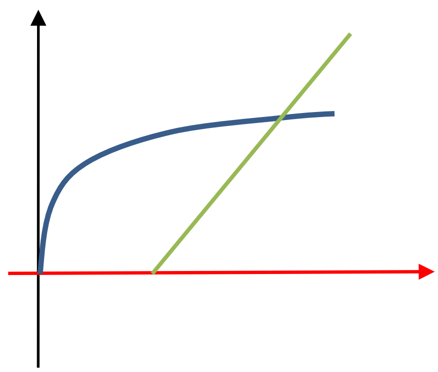
2. Sometimes it is better (or necessary) to set up the integral in *y* instead of in *x*. In this case, we "slice" and add up the rectangles on the vertical, and we need all functions (and bounds) expressed in terms of y.

Example: *Find the area in the 1st quadrant bounded above by $y = \sqrt{x}$ & below by the x-axis and the line $y = x - 2$.*

Solution 1: If we integrate in x, we need to split the region into two pieces:



Solution 2 (much easier!): If we integrate in y, we only need one integral:



NOTE: Sometimes it is useful to know the following facts:

i) If f is an **odd** function (i.e. $f(-x) = -f(x)$, for example: odd powers of x, $\sin x$, $\tan x$, $\csc x$) **and the bounds are symmetric**, the integral $\int_{-a}^{a} f(x)\, dx = 0$

ii) If f is an **even** function (i.e. $f(-x) = f(x)$, for example: even powers of x, $\cos x$, $\sec x$), and **the bounds are symmetric**, then $\int_{-a}^{a} f(x)\, dx = 2\left(\int_0^a f(x)\, dx\right)$

For example: $\displaystyle\int_{-\pi}^{\pi} \frac{x^2 \sin x}{1 + x^2}\, dx = 0$