Math 120AB Winter 2004
Solutions to Final Exam
March 13, 2004

1. An ant is heading toward a circular region of pancake batter. The ant cannot breathe while walking through the batter. The batter has a radius of 4 cm. The ant is going to walk from a point 6 cm due north of the batter's center to a point 5 cm due west of the batter's center.

   If the ant moves at 0.8 cm/sec, how long will it have to hold its breath?

   Solution:

   We set up a coordinate system with the center of circle of batter as the origin. Then the line of the ant's path has slope $\frac{6-0}{0-(-5)} = \frac{6}{5}$. The equation of the line is thus

$$y = \frac{6}{5}x + 6.$$

   The batter is represented by the circle with equation

$$x^2 + y^2 = 4^2 = 16.$$

   We thus seek the intersection of this line and this circle:

$$x^2 + \left(\frac{6}{5}x + 6\right)^2 = 16$$

$$x^2 + \frac{36}{25}x^2 + \frac{72}{5}x + 36 = 16$$

$$\frac{61}{25}x^2 + \frac{72}{5}x + 20 = 0$$

$$x = \frac{-\frac{72}{5} \pm \sqrt{\left(\frac{72}{5}\right)^2 - 4\left(\frac{61}{25}\right)(20)}}{2 \cdot \frac{61}{25}} = -2.236236 \text{ and } -3.665393.$$

   At these values of x, we have

$$y = 3.3165047 \text{ and } y = 1.601528$$

   The distance travelled by the ant through the batter is thus 2.232399 cm, and the time that the ant has to hold its breath is

$$\frac{2.232399 \text{ cm}}{0.8 \text{ cm/sec}} = 2.790499 \text{ sec.}$$

2. The populations of the cities of Alexandria and Springfield are growing exponentially. In 1980, the population of Alexandria was $120,000$ and the population of Springfield was $85,000$. In 1995, the population of Alexandria was $185,000$. The population of Springfield triples every 25 years.

   In what year will there be twice as many people in Springfield as in Alexandria?

   Solution:

   Alexandria:

   Say 1980 is $t = 0$. Then if $A(t)$ is the population (in thousands) of Alexandria t years after 1980, we have

$$
\begin{aligned}
A(t) &= A_0 b^t \\
A(0) = 120 &= A_0 b^0 = A_0 \\
A(15) = 185 &= A_0 b^{15} = 120 b^{15} \\
b^{15} &= \frac{185}{120} \\
b = \left(\frac{185}{120}\right)^{1/15} &= 1.02927802.
\end{aligned}
$$

   So $A(t) = 120(1.02927802)^t$.

   Springfield:

   With a similar setup, we have

$$
\begin{aligned}
S(t) &= C_0 d^t \\
S(0) = 85 &= C_0 d^0 = C_0 \\
S(25) = 3 \cdot 85 &= 85 d^{25} \\
3 &= d^{25} \\
d = 3^{1/25} &= 1.04492435.
\end{aligned}
$$

   So $S(t) = 85(1.04492435)^t$.

   Now, the question asks when there will be twice as many people in Springfield as in Alexandria, so we need to solve the equation

$$
\begin{aligned}
S(t) &= 2A(t) \\
85(1.04492435)^t &= 2 \cdot 120(1.02927802)^t \\
\ln 85 + t \ln 1.04492435 &= \ln 240 + t \ln 1.02927802 \\
t = \frac{\ln 240 - \ln 85}{\ln 1.04492435 - \ln 1.02927802} &= 68.8007
\end{aligned}
$$

   or the year 2048.

3. Suppose the value of my shoehorn collection is growing according to a linear model. In 2000, it was worth $\$4.80$. In 2002, it was worth $\$5.03$.

   My toothpaste collection is also growing in value according to a linear model. In 1990, the collection was worth $\$2.22$, while in 1995, it was worth $\$3.19$.

   When will my toothpaste collection be worth $\$5.00$ more than my shoehorn collection?

   Solution:

   Let the year 2000 be represented by $t = 0$. Then, for shoehorns we have the data points $(0, 4.80)$, and $(2, 5.03)$, so our linear model is

   $$S(t) = 4.80 + 0.115t$$

   For toothpaste, the data points are $(-10, 2.22)$ and $(-5, 3.19)$, so the linear model is

   $$T(t) = 4.16 + 0.194t$$

   The problem thus asks us to solve the equation

   $$T(t) = 5 + S(t)$$

   from which we find

   $$t = 71.3924$$

   or the year 2071.

4. The amount of radiation from a star is observed to be a sinusoidal function of time. You measure the radiation output at its maximum of 400 megawatts at 3 AM one morning. It then drops to its minimum of 180 megawatts at 8 PM that evening. What percentage of the time is the radiation from the star greater than 350 megawatts?

   Solution:

   Using the standard sinusoidal model,

   $$R(t) = A \sin\left(\frac{2\pi}{B}(t - C)\right) + D$$

   we have

   $$A = \frac{400 - 180}{2} = 110$$
   $$B = 2 \cdot 17 = 34$$
   $$D = \frac{400 + 180}{2} = 290$$
   $$C = 3 - \frac{B}{4} = -5.5$$

To answer the question we need first to solve $R(t) = 350$:

$$R(t) = 110\sin\left(\frac{2\pi}{34}(t+5.5)\right) + 290 = 350$$

$$\frac{2\pi}{34}(t+5.5) = \sin^{-1}\frac{60}{110} = 0.5769313$$
$$t = -2.37807.$$

The symmetry solution is

$$3 + (3 - (-2.37807)) = 8.37807.$$

Hence, the time above 350 MW per period is $8.37807 - (-2.37807) = 10.7561$ hours which is

$$\frac{10.7561}{34} = 31.636\%$$

of the time.

5. Susie starts running at a constant speed in a straight line from a point 20 meters due south of a light post in a park to a bench that is 40 meters due east of the light post. It will take her 15 seconds to get to the bench. Express her distance from the light post as a function of the time t since she started running.

   Solution:

   We set up a coordinate system with the light post at the origin, so that her starting point is $(0, -20)$ and the bench is at $(40, 0)$. Since Susie is moving in a straight line at a constant speed, we know that the x- and y-coordinates of her position can be described as linear functions of time. So, for some constants a, b, c, and d, her location $(x(t), y(t))$ at time t is given by

   $$x(t) = a + bt, y(t) = c + dt.$$

   We can take t to be the time since she starts, so that

   $$x(0) = a = 0$$

   and

   $$y(0) = -20 = c$$

   Also,

   $$x(15) = 40 = 0 + 15b$$

   so $b = \dfrac{40}{15}$ and

   $$y(15) = 0 = -20 + 15d$$

so $d = \frac{20}{15}$. Finally then,

$$x(t) = \frac{40}{15}t, \text{ and } y(t) = -20 + \frac{20}{15}t.$$

Thus her distance to the origin (light post) is given by

$$D(t) = \sqrt{\left(\frac{40}{15}t\right)^2 + \left(-20 + \frac{20}{15}t\right)^2}.$$

6. Find all values of d so that the quadratic function

$$f(x) = x(x + d) + 2d$$

has its vertex on the x-axis.

Solution:

We have

$$f(x) = x^2 + dx + 2d = \left(x + \frac{d}{2}\right)^2 - \frac{d^2}{4} + 2d$$

so the y-coordinate of the vertex is

$$y = 2d - \frac{d^2}{4}.$$

For the vertex to be on the x-axis, this must be zero so

$$0 = 2d - \frac{d^2}{4}$$

$$0 = d\left(2 - \frac{d}{4}\right)$$

so $d = 0$ or $d = 8$.

7. Let $g(x) = \frac{3x + 6}{5x - 7}$.

Find $g^{-1}(x)$.

Solution:

Let $y = g(x)$ and solve for x:

$$y = \frac{3x + 6}{5x - 7}$$

$$5yx - 7y = 3x + 6$$

$$(5y-3)x = 6+7y$$
$$x = \frac{6+7y}{5y-3}$$

so

$$g^{-1}(x) = \frac{6+7x}{5x-3}.$$

8. Let $f(x) = \dfrac{(x+3)(2x-5)}{x^2-5x-6}$.

   (a) Find the zeros of $f(x)$.
   (b) Find the horizontal asymptote of $f(x)$ or state that it has none.
   (c) Find all vertical asymptotes of $f(x)$.

   Solution:

   (a) If $f(x) = 0$, then
   $$(x+3)(2x-5) = 0$$
   so $x = -3$ or $x = \dfrac{5}{2}$. Since the denominator factors as
   $$(x-6)(x+1)$$
   we see that the denominator is not made zero by these values, so they are the zeros of $f(x)$.

   (b) We may write
   $$f(x) = \frac{(x+3)(2x-5)}{x^2-5x-6} = \frac{2x^2+x-15}{x^2-5x-6}\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right) = \frac{2+\frac{1}{x}-\frac{15}{x^2}}{1-\frac{5}{x}-\frac{6}{x^2}} \approx \frac{2+0-0}{1-0-0} = 2$$
   when x is large. Thus, $y = 2$ is the horizontal asymptote.

   (c) Since the denominator of $f(x)$ factors as
   $$(x-6)(x+1)$$
   we see that the function has vertical asymptotes at $x = 6$ and $x = -1$ (noting that the numerator is not made zero by these values).

9. Patrick is riding his bicycle around a circular track. The track has a radius of 50 meters.

(a) If a ray from the center of the track to Patrick sweeps out area at the rate of 150 square meters per second, how fast is Patrick moving?
Solution:
There are many ways to solve this problem. One is to note that the area of the entire circle swept out by the ray is

$$\pi(50^2) = 7853.98163 \text{ square meters.}$$

Since the ray sweeps out 150 square meters per second, it will take

$$\frac{7853.98163 \text{ square meters}}{150 \text{ square meters per second}} = 52.359878 \text{ seconds}$$

to sweep out the whole circle. That is, Patrick does a lap of the track every 52.359878 seconds. Hence, his angular speed is

$$\frac{2\pi}{52.359878} = 0.12 \text{ radians per second}$$

and so his linear speed is

$$(0.12 \text{ radians per second})(50 \text{ meters}) = 6 \text{ meters per second.}$$

(b) Patrick's coach is standing at the edge of the track. Patrick passes her while riding at a constant 11 meters per second. What is the straight-line distance between Patrick and his coach 12 seconds later?

Solution:

Setting the origin at the center of the track and the coach at the point $(50, 0)$, Patrick's position t seconds after passing the coach is given by

$$(50\cos(\omega t), 50\sin(\omega t))$$

where ω is Patrick's angular speed. Since his linear speed is 11 meters per second, his angular speed can be found using the relation $v = r\omega$, i.e.,

$$\omega = \frac{v}{r} = \frac{11 \text{ m/sec}}{50 \text{ m}} = 0.22 \text{ radians per second.}$$

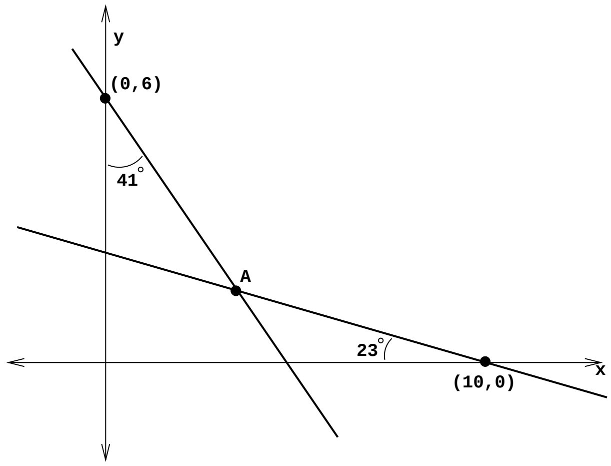
Thus, after 12 seconds, Patrick's location is

$$(50\cos(0.22 \cdot 12), 50\sin(0.22 \cdot 12)) = (-43.8409, 24.0411)$$

and so the distance from him to his coach is

$$\sqrt{(50 + 43.8409)^2 + 24.0411^2} = 96.8715 \text{ meters.}$$

10. Find the coordinates of the point labelled A in the figure below.



Solution:

The line passing through $(0, 6)$ intersects the x-axis at a point with x-coordinate a with

$$\frac{a}{6} = \tan 41^\circ$$

i.e., $a = 5.2157204$. Hence the line has slope

$$-\frac{6}{5.2157204} = -1.150368$$

and equation $y = -1.150368x + 6$.

The other line intersects the y-axis at the point $(0, b)$ with

$$\frac{b}{10} = \tan 23^\circ$$

so $b = 10 \tan 23^\circ = 4.24474816$ and the line has equation

$$y = -0.424474816x + 4.24474816$$

Setting these two lines equal to each other and solving we find A has coordinates

$$x = 2.4180588, y = 3.218342.$$