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A ZETA FUNCTION FOR $\mathbb{Z}^d$-ACTIONS

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ABSTRACT. We define a zeta function for $\mathbb{Z}^d$-actions α that generalizes the Artin-Mazur zeta function for a single transformation. This zeta function is a conjugacy invariant, can be computed explicitly in some cases, and has a product formula over finite orbits. The analytic behavior of the zeta function for $d \geq 2$ is quite different from the case $d = 1$. Even for higher-dimensional actions of finite type the zeta function is typically transcendental and has natural boundary a circle of finite radius. We compute the radius of convergence of the zeta function for a class of algebraic $\mathbb{Z}^d$-actions. We conclude by conjecturing a general description of the analytic behavior of these zeta functions and discussing some further problems.

1. INTRODUCTION

Let $\phi\colon X \to X$ be a homeomorphism of a compact space and $p_n(\phi)$ denote the number of points in X fixed by ϕ^n. We assume that $p_n(\phi)$ is finite for all $n \geq 1$. Artin and Mazur [1] introduced a zeta function $\zeta_\phi(s)$ for ϕ defined by

$$\zeta_\phi(s) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n(\phi)}{n} s^n\right). \tag{1.1}$$

Bowen and Lanford [2] showed that if ϕ is a shift of finite type, then $\zeta_\phi(s)$ is a rational function. This was extended to the case when ϕ is an Axiom A diffeomorphism by Manning [11] and to finitely presented systems by Fried [4]. If ϕ is expansive then an elementary argument shows that $\zeta_\phi(s)$ has radius of convergence at least $\exp(-h(\phi))$, where $h(\phi)$ is the topological entropy of ϕ. Finally, the zeta function has the product formula

$$\zeta_\phi(s) = \prod_\gamma \frac{1}{1 - s^{|\gamma|}}, \tag{1.2}$$

where the product is over all finite orbits γ of ϕ and $|\gamma|$ denotes the number of points in γ.

The purpose of this paper is to define an analogous zeta function for $\mathbb{Z}^d$-actions generated by d commuting homeomorphisms. First some notation. Let α be an action of $\mathbb{Z}^d$ on X. For $\mathbf{n} \in \mathbb{Z}^d$ let $\alpha^{\mathbf{n}}$ denote the element of this action corresponding to $\mathbf{n}$. Denote the set of finite-index subgroups of $\mathbb{Z}^d$ by

Supported in part by NSF Grant DMS 9303240

$\mathcal{L}_d$. For $L \in \mathcal{L}_d$ let $p_L(\alpha)$ be the number of points in X fixed by $\alpha^{\mathbf{n}}$ for all $\mathbf{n} \in L$. The index $|\mathbb{Z}^d/L|$ of L in $\mathbb{Z}^d$ is denoted by $[L]$. Note that when $d=1$ we have $\mathcal{L}_1 = \{n\mathbb{Z} : n \geq 1\}$. We generalize (1.1) to $d \geq 1$ by replacing the sum over $n \geq 1$ by the sum over $L \in \mathcal{L}_d$. This leads to our definition of the zeta function of α as

$$\zeta_\alpha(s) = \exp\biggl(\sum_{L\in\mathcal{L}_d} \frac{p_L(\alpha)}{[L]} s^{[L]}\biggr). \tag{1.3}$$

In this paper we establish some basic properties of the zeta function of a $\mathbb{Z}^d$-action, and compute this function explicitly for a number of examples. Even the trivial $\mathbb{Z}^2$-action on a point has the "interesting" zeta function

$$\prod_{n=1}^{\infty} \frac{1}{1-s^n} = \sum_{n=1}^{\infty} p(n)\, s^n,$$

where $p(n)$ is the number of partitions of n. This formula goes back to Euler, and was the starting point for proving the asymptotic formula

$$p(n) \sim \exp\bigl(\pi\sqrt{2n/3}\bigr)$$

developed by Hardy and Ramanujan [5]. We prove an analogue of the product formula (1.2) in §5, and provide evidence for the conjecture that when $d \geq 2$ the zeta function is typically meromorphic for $|s| < \exp(-h(\alpha))$ and has the circle $|s| = \exp(-h(\alpha))$ as its natural boundary.

Another approach to generalizing the zeta function to $\mathbb{Z}^d$-actions was taken by Mathiszik in his dissertation [12], which was never published. There he counts multiples of the period of a point, not the lattices contained in the stabilizer of the periodic point, leading to a different function. The definition (1.3) was stated independently by Ward in the preprint [16].

2. Definition and basic properties

Let α be a $\mathbb{Z}^d$-action on X, and $\mathcal{L}_d$ denote the collection of finite-index subgroups (or *lattices*) in $\mathbb{Z}^d$. For $L \in \mathcal{L}_d$ put $[L] = |\mathbb{Z}^d/L|$, and let

$$p_L(\alpha) = |\{x \in X : \alpha^{\mathbf{n}}x = x \text{ for all } \mathbf{n} \in L\}|,$$

which we will assume to be finite for all $L \in \mathcal{L}_d$.

Definition 2.1. The *zeta function* of the $\mathbb{Z}^d$-action α is defined as

$$\zeta_\alpha(s) = \exp\biggl(\sum_{L\in\mathcal{L}_d} \frac{p_L(\alpha)}{[L]} s^{[L]}\biggr).$$

When $d=1$ note that $\mathcal{L}_1 = \{n\mathbb{Z} : n \geq 1\}$ and that $p_{n\mathbb{Z}}(\alpha) = p_n(\phi)$, where $\phi = \alpha^1$ is the generator of α. Hence the zeta function of a $\mathbb{Z}$-action generated by ϕ is just the Artin-Mazur zeta function of ϕ.

The zeta function is obviously a conjugacy invariant for $\mathbb{Z}^d$-actions.

For $A \in GL(d,\mathbb{Z})$ define the $\mathbb{Z}^d$-action α^A by $(\alpha^A)^{\mathbf{n}} = \alpha^{A\mathbf{n}}$. The following shows that the zeta function is independent of a choice of basis for $\mathbb{Z}^d$.

Lemma 2.2. Let α be a $\mathbb{Z}^d$-action and $A \in GL(d,\mathbb{Z})$. Then $\zeta_\alpha(s) = \zeta_{\alpha^A}(s)$.

Proof. For $L \in \mathcal{L}_d$ let $AL = \{A\mathbf{n} : \mathbf{n} \in L\}$. Then $L \longleftrightarrow AL$ is a bijection from $\mathcal{L}_d$ to itself, and $[AL] = [L]$. Clearly

$$p_L(\alpha^A) = p_{AL}(\alpha),$$

and the result now follows. □

If $Y \subset X$ is α-invariant, let $\alpha|Y$ denote the restriction of α to Y.

Lemma 2.3. Let $Y, Z \subset X$ be compact α-invariant sets. Then

$$\zeta_{\alpha|Y\cup Z}(s) = \frac{\zeta_{\alpha|Y}(s)\zeta_{\alpha|Z}(s)}{\zeta_{\alpha|Y\cap Z}(s)}.$$

In particular, if Y and Z are disjoint, then

$$\zeta_{\alpha|Y\cup Z}(s) = \zeta_{\alpha|Y}(s)\zeta_{\alpha|Z}(s).$$

Proof. For $L \in \mathcal{L}_d$ it is elementary that

$$p_L(\alpha|Y\cup Z) = p_L(\alpha|Y) + p_L(\alpha|Z) - p_L(\alpha|Y\cap Z),$$

and the result follows from the definition of the zeta function. □

3. Examples

In this section we compute the zeta function for some specific $\mathbb{Z}^2$-actions. For this we require a parameterization of $\mathcal{L}_2$. A convenient one for our purposes is

$$\mathcal{L}_2 = \left\{\begin{bmatrix} a & b \\ 0 & c\end{bmatrix}\mathbb{Z}^2 : a \geq 1,\ c \geq 1,\ 0 \leq b \leq a-1\right\}. \tag{3.1}$$

The Hermite normal form for integral matrices, discussed in the next section, shows that this gives a complete listing of the lattices in $\mathbb{Z}^2$, with each lattice listed exactly once.

We also use the familiar power series

$$-\log(1-t) = \sum_{n=1}^{\infty} \frac{t^n}{n}. \tag{3.2}$$

Example 3.1. Let α be the trivial $\mathbb{Z}^2$-action on a single point. Then $p_L(\alpha) = 1$ for all $L \in \mathcal{L}_2$. Hence using the parameterization (3.1) of $\mathcal{L}_2$ we see that

$$\begin{aligned}\zeta_\alpha(s) &= \exp\left(\sum_{a=1}^{\infty}\sum_{c=1}^{\infty}\sum_{b=0}^{a-1}\frac{1}{ac}s^{ac}\right) = \exp\left(\sum_{a=1}^{\infty}\sum_{c=1}^{\infty}\frac{(s^a)^c}{c}\right)\\ &= \exp\left(\sum_{a=1}^{\infty} -\log(1-s^a)\right)\\ &= \prod_{a=1}^{\infty}\frac{1}{1-s^a}.\end{aligned}$$

We denote this zeta function by $\pi_2(s)$. As noted in §1, $\pi_2(s)$ is the classical generating function for the partition function, i.e.,

$$\pi_2(s) = \sum_{n=1}^{\infty} p(n)\, s^n.$$

It is known that this function is analytic in $|s| < 1$ and has the unit circle $|s| = 1$ as its natural boundary [5].

Example 3.2. Let α be the full $\mathbb{Z}^2$ k-shift. This means that $X = \{0, 1, \dots, k-1\}^{\mathbb{Z}^2}$ and α is the natural shift action of $\mathbb{Z}^2$ on X. Suppose $L \in \mathcal{L}_2$. Then a point in X that is fixed by all $\alpha^{\mathbf{n}}$ for $\mathbf{n} \in L$ is determined by its coordinates in a fundamental domain for L, and all choices there are possible. Hence

$$p_L(\alpha) = k^{[L]}.$$

Since $p_L(\alpha)$ depends only on the index of L, a calculation similar to Example 3.1 shows that here

$$\zeta_\alpha(s) = \prod_{a=1}^{\infty}\frac{1}{1-(ks)^a} = \pi_2(ks).$$

Note that $h(\alpha) = \log k$, so that $\zeta_\alpha(s)$ is analytic in $|s| < \exp(-h(\alpha)) = 1/k$ and has the circle $|s| = \exp(-h(\alpha))$ as its natural boundary.

Example 3.3. Let $X = \{0, 1, \dots, k-1\}^{\mathbb{Z}}$, and σ be the shift $\mathbb{Z}$-action on X. Define the $\mathbb{Z}^2$-action α on X by $\alpha^{(m,n)} = \sigma^n$, so that α is generated by the identity map and σ. This $\mathbb{Z}^2$-action is clearly conjugate to the $\mathbb{Z}^2$ shift action β on

$$Y = \left\{y \in \{0, 1, \dots, k-1\}^{\mathbb{Z}^2} : y_{i,j} = y_{i+1,j} \text{ for all } i, j \in \mathbb{Z}\right\}.$$

We will compute $\zeta_\beta(s) = \zeta_\alpha(s)$.

Let

$$L = \begin{bmatrix} a & b \\ 0 & c\end{bmatrix}\mathbb{Z}^2,$$

where $a \ge 1$, $c \ge 1$, and $0 \le b \le a-1$. A point $y \in Y$ that is fixed by $\beta^{\mathbf{n}}$ for all $\mathbf{n} \in L$ is determined by its coordinates $y_{0,0}, y_{0,1}, \dots, y_{0,c-1}$, and the choices for these are arbitrary. Hence $p_L(\beta) = k^c$. Thus

$$\begin{aligned}\zeta_\alpha(s) = \zeta_\beta(s) &= \exp\left(\sum_{a=1}^{\infty}\sum_{c=1}^{\infty}\sum_{b=0}^{a-1}\frac{k^c}{ac}s^{ac}\right)\\ &= \exp\left(\sum_{a=1}^{\infty}\sum_{c=1}^{\infty}\frac{(ks^a)^c}{c}\right)\\ &= \exp\left(\sum_{a=1}^{\infty} -\log(1-ks^a)\right)\\ &= \prod_{a=1}^{\infty}\frac{1}{1-ks^a} = \frac{1}{(1-ks)(1-ks^2)(1-ks^3)\dots}.\end{aligned}$$

Observe that $h(\alpha) = 0$, but the factor $1 - ks$ in the denominator shows that $\zeta_\alpha(s)$ has a simple pole at $1/k$. This shows that the radius of convergence of ζ_α may be strictly smaller that $\exp(-h(\alpha))$. In fact, the product development above for $\zeta_\alpha(s)$ shows that it has poles at

$$\left\{\frac{1}{\sqrt[n]{k}}e^{2\pi ij/n} : 0 \le j \le n-1,\ n \ge 1\right\}.$$

Hence $\zeta_\alpha(s)$ has infinitely many poles inside the unit disk, and these poles cluster to the unit circle. It follows that $\zeta_\alpha(s)$ is meromorphic in $|s| < \exp(-h(\alpha)) = 1$ and has natural boundary $|s| = \exp(-h(\alpha)) = 1$.

Example 3.4. Let $\mathbb{F}_2 = \{0, 1\}$ be the field with two elements, and

$$X = \left\{x \in \mathbb{F}_2^{\mathbb{Z}^2} : x_{i,j} + x_{i+1,j} + x_{i,j+1} = 0 \text{ for all } i, j \in \mathbb{Z}\right\}.$$

Then X is a compact group with coordinate-wise operations, and it is invariant under the natural $\mathbb{Z}^2$ shift action α. This action was originally investigated by Ledrappier [8], who showed that it was mixing, but not mixing of higher orders.

Let

$$L = \begin{bmatrix} a & b \\ 0 & c\end{bmatrix}\mathbb{Z}^2,$$

where $a \ge 1$, $c \ge 1$, and $0 \le b \le a-1$. We compute $p_L(\alpha)$ using some linear algebra over $\mathbb{F}_2$. A point $x \in X$ that is L-invariant must have horizontal period a, so is determined by an element y from the vector space $\mathbb{F}_2^a$. Let I_a denote the $a \times a$ identity matrix and P_a the $a \times a$ permutation matrix corresponding to the cyclic shift of elementary basis vectors of $\mathbb{F}_2^a$. The condition of L-periodicity of x translates to the condition

$$(I_a + P_a)^c\, y = P_a^{-b}\, y.$$

Hence

$$p_L(\alpha) = \left|\ker\left((I_a + P_a)^c - P_a^{-b}\right)\right|. \tag{3.3}$$

For example, if $a = 3$, $b = 1$, and $c = 1$, then

$$P_a = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$(I_a + P_a)^c - P_a^{-b} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The kernel of the latter has dimension two over $\mathbb{F}_2$, so that for this L we have $p_L(\alpha) = 4$.

Formula (3.3) allows computation of $\zeta_\alpha(s)$ to as many terms as we like. A calculation using *Mathematica* gives the first terms to be

$$\zeta_\alpha(s) = 1 + s + 2s^2 + 4s^3 + 6s^4 + 9s^5 + 16s^6 + 24s^7 + 35s^8 + 54s^9 + 78s^{10} + 110s^{11} + 162s^{12} + 226s^{13} + 317s^{14} + 446s^{15} + 612s^{16} + \dots.$$

Such calculations do not, however, reveal the analytic properties of $\zeta_\alpha(s)$. For this we next employ other ideas to show that $\zeta_\alpha(s)$ has radius of convergence one and the unit circle as natural boundary.

We first estimate the size of $p_L(\alpha)$ in terms of $[L]$.

Lemma 3.5. Let α be the $\mathbb{Z}^2$ shift action on

$$X = \{x \in \mathbb{F}_2^{\mathbb{Z}^2} : x_{i,j} + x_{i+1,j} + x_{i,j+1} = 0 \quad \text{for all } i, j \in \mathbb{Z}\}.$$

Then there is a constant C such that for all $L \in \mathcal{L}_2$ we have that

$$p_L(\alpha) \le 2^{C\sqrt{[L]}}. \tag{3.4}$$

Proof. Let $L \in \mathcal{L}_2$. We first show that there is a nonzero vector $\mathbf{v} \in L$ such that $\|\mathbf{v}\| \le \theta\sqrt{[L]}$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^2$ and $\theta = \sqrt{4/\pi}$. Let B_r be the ball of radius r in $\mathbb{R}^2$ around $\mathbf{0}$, and A be an integral matrix such that $A\mathbb{Z}^2 = L$. If $B_r \cap L = \{\mathbf{0}\}$, then $A^{-1}(B_r)$ is a convex, symmetric region in $\mathbb{R}^2$ that does not contain any nonzero vectors in $\mathbb{Z}^2$. By Minkowski's theorem [6, Thm. 37] we have that

$$\text{area}\left(A^{-1}(B_r)\right) < 4.$$

Since $\det A = [L]$, it follows that

$$\frac{\pi r^2}{[L]} < 4, \quad \text{or} \quad r < \theta\sqrt{[L]}.$$

Hence the ball of radius $\theta\sqrt{[L]}$ must contain a nonzero vector $\mathbf{v}$ in L.

Next, consider the strip S of width 2 about the line segment $[\mathbf{0}, \mathbf{v}]$. If $x \in X$ is L-periodic, then its coordinates in S determine by periodicity those in the strip of width 2 about the line $\mathbb{R}\mathbf{v}$, which in turn determine the rest of the coordinates by use of the defining relations for points in X and L-periodicity. Hence the number of L-periodic points in X is bounded above by the number of possible configurations in S. Since the the number of lattice points in S is bounded by a constant times $\sqrt{[L]}$, the estimate (3.4) now follows. □

Continuing with Example 3.4, the parameterization (3.1) of $\mathcal{L}_2$ shows that the number of $L \in \mathcal{L}_2$ with $[L] = n$ equals $\sigma(n)$, the sum of the divisors of n. Trivially $\sigma(n) \le 1 + 2 + \cdots + n < n^2$, so that

$$\sum_{\{L \in \mathcal{L}_2 : [L] = n\}} p_L(\alpha) \le n^2 \cdot 2^{C\sqrt{n}}.$$

It follows from the Hadamard formula that the series for $\zeta_\alpha(s)$ has radius of convergence one.

We next show that $\zeta_\alpha(s)$ has the unit circle as natural boundary. For this, we first claim that the estimate (3.4) is sharp in the sense that no power of $[L]$ strictly less than $1/2$ will work for all L. Let

$$L_n = (2^n - 1)\mathbb{Z} \oplus (2^n - 1)\mathbb{Z},$$

so that $[L_n] = (2^n - 1)^2$. The expansion of $(1 + t)^{2^n - 1}$ mod 2 is $\sum_{j=0}^{2^n - 1} t^j$. It then follows from (3.3) that $p_{L_n}(\alpha)$ equals the number of elements in $\mathbb{F}_2^{2^n - 1}$ whose coordinates sum to 0, which is half the cardinality of $\mathbb{F}_2^{2^n - 1}$. Hence

$$p_{L_n}(\alpha) = \frac{1}{2}\, 2^{2^n - 1} = \frac{1}{2}\, 2^{\sqrt{[L_n]}}, \tag{3.5}$$

verifying our claim.

Suppose that $\zeta_\alpha(s)$ were rational, say

$$\zeta_\alpha(a) = \frac{\prod_{i=1}^{k}(1 - \lambda_i s)}{\prod_{j=1}^{l}(1 - \mu_j s)}.$$

It follows from calculus that we would then have

$$\sum_{\{L \in \mathcal{L}_2 : [L] = n\}} p_L(\alpha) = \sum_{l=1}^{l} \mu_j^n - \sum_{i=1}^{k} \lambda_i^n. \tag{3.6}$$

Now $\zeta_\alpha(s)$ is analytic in $|s| < 1$, so that $|\mu_j| \le 1$ for all j. Furthermore, $\zeta_\alpha(s)$ does not vanish in $|s| < 1$, being the exponential of a convergent power series there, so that $|\lambda_i| \le 1$ for all i. Thus (3.6) would imply that $p_L(\alpha) \le k + l$ for all $L \in \mathcal{L}_2$, contradicting (3.5). This proves that $\zeta_\alpha(s)$ is not rational.

The product formula, which is proved in §5, shows that the Taylor series of $\zeta_\alpha(s)$ has integer coefficients (see Corollary 5.5). A theorem of Carlson [3] (cf. Polya [13]) asserts that an analytic function whose Taylor series has integer coefficients and radius of convergence one is either rational or has the unit circle as natural boundary. For $\zeta_\alpha(s)$ we have just ruled out the first alternative. Hence $\zeta_\alpha(s)$ is analytic in $|s| < 1$ and has the circle $|s| = 1$ as its natural boundary.

4. Calculation for trivial $\mathbb{Z}^d$-actions

In this section we explicitly compute the zeta function $\pi_d(s)$ of the trivial $\mathbb{Z}^d$-action on a point. For $d = 1$, equation (3.2) shows that

$$\pi_1(s) = \frac{1}{1-s}.$$

In Example 3.1 we showed that

$$\pi_2(s) = \prod_{n=1}^{\infty} \frac{1}{1-s^n}.$$

The functions $\pi_d(s)$ play a central role in the product formula in §5.

Let α be the trivial $\mathbb{Z}^d$-action on a point. Then $p_L(\alpha) = 1$ for all $L \in \mathcal{L}_d$. Hence

$$\pi_d(s) = \zeta_\alpha(s) = \exp\left(\sum_{L \in \mathcal{L}_d} \frac{1}{[L]} s^{[L]} \right) = \exp\left(\sum_{n=1}^{\infty} \frac{e_d(n)}{n} s^n \right),$$

where

$$e_d(n) = \left| \{ L \in \mathcal{L}_d : [L] = n \} \right|. \tag{4.1}$$

To compute $e_d(n)$ we use the *Hermite normal form* of an integer matrix (see [10, Thm. 22.1]).

Theorem 4.1 (Hermite). Every lattice in $\mathcal{L}_d$ has a unique representation as the image of $\mathbb{Z}^d$ under a matrix having the form

$$\begin{bmatrix} a_1 & b_{12} & b_{13} & \dots & b_{1d} \\ 0 & a_2 & b_{23} & \dots & b_{2d} \\ 0 & 0 & a_3 & \dots & b_{3d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_d \end{bmatrix}, \tag{4.2}$$

where $a_i \geq 1$ for $1 \leq i \leq d$ and $0 \leq b_{ij} \leq a_i - 1$ for $i + 1 \leq j \leq d$. □

This result provides a convenient parameterization of $\mathcal{L}_d$, generalizing that of $\mathcal{L}_2$ we described in equation (3.1). It also shows how to inductively compute the $e_d(n)$ as follows.

	$n=1$	2	3	4	5	6	7	8	9	10
$d=1$	1	1	1	1	1	1	1	1	1	1
2	1	3	4	7	6	12	8	15	13	18
3	1	7	13	35	31	91	57	155	130	217
4	1	15	40	155	156	600	400	1395	1210	2340
5	1	31	121	651	781	3751	2801	11811	11011	24211
6	1	63	364	2667	3906	22932	19608	97155	99463	246078

Table 1. Values of $e_d(n)$

Proposition 4.2. Let $e_d(n)$ be the number of lattices in $\mathbb{Z}^d$ having index n. Then

$$e_d(n) = \sum_{k|n} e_{d-1}\left(\frac{n}{k}\right) k^{d-1}. \tag{4.3}$$

Proof. Suppose that $L \in \mathcal{L}_d$ is the image of $\mathbb{Z}^d$ under the matrix in (4.2). Then $[L] = a_1 a_2 \dots a_{d-1} a_d$, so that a_1 divides $[L]$. Let $k = a_1$. Each of the $b_{12}, b_{13}, \dots, b_{1d}$ can assume the values $0, 1, \dots, k-1$, giving k^{d-1} choices for the top row. There are $e_{d-1}(n/k)$ choices for the remaining part of the matrix. Summing over all divisors k of n gives (4.3). □

For example, we have trivially that

$$e_0(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

Then (4.3) with $d = 1$ gives $e_1(n) = 1$ for all $n \geq 1$, and next with $d = 2$ that $e_2(n) = \sigma(n)$, the sum of the divisors of n. Table 1 lists some values of $e_d(n)$.

It follows from (4.3) that the sequence $\{e_d(n) : n \geq 1\}$ is the Dirichlet product of $\{e_{d-1}(n) : n \geq 1\}$ with the sequence $\{n^{d-1} : n \geq 1\}$. Inductively this shows that $\{e_d(n) : n \geq 1\}$ is the Dirichlet product of the sequences $\{n^0\}$, $\{n^1\}, \dots, \{n^{d-1}\}$. Now the Dirichlet generating function of $\{n^k\}$ is

$$\sum_{n=1}^{\infty} \frac{n^k}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^{s-k}} = \zeta(s-k),$$

where $\zeta(s)$ denotes the Riemann zeta function (see [6, §17.5]). Since the Dirichlet generating function of the Dirichlet product of sequences is the product of their Dirichlet generating functions, we see that

$$\sum_{n=1}^{\infty} \frac{e_d(n)}{n^s} = \zeta(s)\zeta(s-1)\dots\zeta(s-d+1).$$

This provides an explicit representation of the $e_d(n)$.

We next estimate the growth rate of $e_d(n)$ as $n \to \infty$.

Proposition 4.3. Let $e_d(n)$ be defined as in (4.1). Then

$$e_d(n) \leq n^{d+1}. \tag{4.4}$$

Proof. Since $e_1(n) = 1$ for all $n \geq 1$, the estimate (4.4) is certainly valid for $d = 1$. Assume inductively that it holds when $d \geq 2$ is replaced by $d-1$. Then by (4.3),

$$e_d(n) = \sum_{k|n} e_{d-1}\left(\frac{n}{k}\right) k^{d-1} \leq \sum_{k|n} \left(\frac{n}{k}\right)^d k^{d-1} \leq n^d \sum_{k=1}^{n} \frac{1}{k} \leq n^{d+1},$$

completing the inductive step and the proof. □

To compute $\pi_d(s)$, we make use of the following observation.

Lemma 4.4. Let β be a $\mathbb{Z}^{d-1}$-action and α be the $\mathbb{Z}^d$-action defined by

$$\alpha^{(n_1,n_2,\ldots,n_d)} \doteq \beta^{(n_2,n_3,\ldots,n_d)}.$$

Then

$$\zeta_\alpha(s) = \prod_{k=1}^{\infty} \zeta_\beta(s^k)^{k^{d-2}}.$$

Proof. Consider a $(d-1) \times (d-1)$ matrix

$$B = \begin{bmatrix} a_2 & b_{23} & \ldots & b_{2d} \\ 0 & a_3 & \ldots & b_{3d} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_d \end{bmatrix}$$

in Hermite normal form. We can form all $d \times d$ matrices A in Hermite normal form containing B as the lower right principal minor by appending to B a top row of the form $[k \; b_{12} \; b_{13} \; \ldots \; b_{1d}]$, where $k \geq 1$ and $0 \leq b_{1j} \leq k-1$, and making the remaining entries in the left column 0, forming

$$A = \begin{bmatrix} k & b_{12} & b_{13} & \ldots & b_{1d} \\ 0 & a_2 & b_{23} & \ldots & b_{2d} \\ 0 & 0 & a_3 & \ldots & b_{3d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_d \end{bmatrix}.$$

Note that since $\alpha^{(1,0,\ldots,0)}$ is the identity, we have that

$$p_{A\mathbb{Z}^d}(\alpha) = p_{B\mathbb{Z}^{d-1}}(\beta)$$

for all choices of k and the b_{1j}. Hence

$$\begin{aligned} \zeta_\alpha(s) &= \exp\left(\sum_{M \in \mathcal{L}_{d-1}} \sum_{k=1}^{\infty} \sum_{b_{12}=0}^{k-1} \cdots \sum_{b_{1d}=0}^{k-1} \frac{p_M(\beta)}{k[M]} s^{k[M]}\right) \\ &= \exp\left(\sum_{M \in \mathcal{L}_{d-1}} \sum_{k=1}^{\infty} \frac{k^{d-1} p_M(\beta)}{k[M]} s^{k[M]}\right) \\ &= \prod_{k=1}^{\infty} \exp\left(\sum_{M \in \mathcal{L}_{d-1}} \frac{p_M(\beta)}{[M]} (s^k)^{[M]}\right)^{k^{d-2}} \\ &= \prod_{k=1}^{\infty} \zeta_\beta(s^k)^{k^{d-2}}. \qquad \square \end{aligned}$$

This lemma implies, for example, that if α is the $\mathbb{Z}^2$-action generated by the identity on X and $\phi: X \to X$, then

$$\zeta_\alpha(s) = \prod_{k=1}^{\infty} \zeta_\phi(s^k).$$

We used this in Example 3.1 to compute that

$$\pi_2(s) = \prod_{k=1}^{\infty} \pi_1(s^k) = \prod_{k=1}^{\infty} \frac{1}{1-s^k},$$

and also in Example 3.3. A general version of this argument leads to the following formula for $\pi_d(s)$.

Theorem 4.5. Let $\pi_d(s)$ denote the zeta function of the trivial $\mathbb{Z}^d$-action on a point. Then the Taylor series of $\pi_d(s)$ has radius of convergence one. Furthermore, $\pi_d(s)$ has the product expansion

$$\pi_d(s) = \prod_{n=1}^{\infty} \frac{1}{(1-s^n)^{e_{d-1}(n)}}, \tag{4.5}$$

where $e_d(n)$ is defined in (4.1) and the product converges for $|s| < 1$.

Proof. By Proposition 4.3 we know that $1 \leq e_d(n) \leq n^{d+1}$. Hence the series

$$\psi(s) = \sum_{n=1}^{\infty} \frac{e_d(n)}{n} s^n$$

has radius of convergence one. Furthermore, $\lim_{s \to 1^-} \psi(s) = \infty$. Hence the Taylor series of $\pi_d(s) = e^{\psi(s)}$ has radius of convergence one.

We have already seen that (4.5) is valid for $d = 2$. Suppose inductively that it is valid when $d \geq 3$ is replaced by $d-1$. Lemma 4.4 applies to the trivial $\mathbb{Z}^d$-action α and the trivial $\mathbb{Z}^{d-1}$-action β. Hence using the lemma and Proposition 4.2, we obtain

$$\pi_d(s) = \prod_{k=1}^{\infty} \pi_{d-1}(s^k)^{k^{d-2}} = \prod_{k=1}^{\infty}\prod_{n=1}^{\infty}(1-s^{kn})^{-e_{d-2}(n)\,k^{d-2}}$$
$$= \prod_{m=1}^{\infty}(1-s^m)^{-\sum_{k|m} e_{d-2}(m/k)\,k^{d-2}} = \prod_{m=1}^{\infty}(1-s^m)^{-e_{d-1}(m)},$$

where the estimate (4.4) shows that our manipulations are valid when $|s| < 1$. This verifies (4.5) for d, completing the proof. □

Proposition 4.6. For $d \geq 2$ the function $\pi_d(s)$ has the unit circle as natural boundary.

Proof. Theorem 4.5 shows that the Taylor series for $\pi_d(s)$ has integer coefficients and radius of convergence one. Also, an easy consequence of Proposition 4.2 is that $e_d(n) \geq n$ for $d \geq 2$, so that $e_d(n) \to \infty$ as $n \to \infty$. The same argument as at the end of Example 3.4 shows that $\pi_d(s)$ is not a rational function for $d \geq 2$, and the same use of Carlson's theorem as there then proves that $\pi_d(s)$ has the unit circle as natural boundary. □

5. The product formula

Our goal is this section is to generalize the product formula (1.2) for single transformations to $\mathbb{Z}^d$-actions α. We begin by considering the case when α has exactly one finite orbit.

Proposition 5.1. Let γ be a finite set and α be a transitive $\mathbb{Z}^d$-action on γ. Then

$$\zeta_\alpha(s) = \pi_d(s^{|\gamma|}).$$

Proof. Let $H = \{\mathbf{n} \in \mathbb{Z}^d : \alpha^{\mathbf{n}}x = x \text{ for all } x \in \gamma\}$. Since γ is finite, $H \in \mathcal{L}_d$. Now α acts transitively on γ, so that H is also the stabilizer subgroup of each element of γ, so that $|\gamma| = [H]$. Hence for $L \in \mathcal{L}_d$ we have that

$$p_L(\alpha) = \begin{cases} |\gamma| & \text{if } L \subseteq H, \\ 0 & \text{if } L \not\subseteq H. \end{cases}$$

Since $[H]/[L] = 1/|H/L|$, we see that

$$\zeta_\alpha(s) = \exp\left(\sum_{\{L\in\mathcal{L}_d:\, L\subseteq H\}} \frac{[H]}{[L]}\, s^{[L]}\right)$$
$$= \exp\left(\sum_{\{L\in\mathcal{L}_d:\, L\subseteq H\}} \frac{1}{|H/L|}\, (s^{[H]})^{|H/L|}\right)$$
$$= \pi_d(s^{[H]}) = \pi_d(s^{|\gamma|}). \quad \square$$

In order to determine the validity of the product formula, we need to first find the radius of convergence of the Taylor series for $\zeta_\alpha(s)$. To do this, we introduce the following quantity.

Definition 5.2. Let α be a $\mathbb{Z}^d$-action. Define the *growth rate of periodic points of* α as

$$g(\alpha) = \limsup_{[L]\to\infty} \frac{1}{[L]} \log p_L(\alpha) = \lim_{n\to\infty} \sup_{[L]\geq n} \frac{1}{[L]} \log p_L(\alpha).$$

Theorem 5.3. Let α be a $\mathbb{Z}^d$-action, and assume as usual that $p_L(\alpha)$ is finite for every $L \in \mathcal{L}_d$. Then $\zeta_\alpha(s)$ has radius of convergence $\exp(-g(\alpha))$. In particular, if there is a number $\theta > 1$ for which $p_L(\alpha) \leq \theta^{[L]}$ for all $L \in \mathcal{L}_d$, then $\zeta_\alpha(s)$ is analytic for $|s| < \theta^{-1}$.

Proof. By Proposition 4.3, the number of $L \in \mathcal{L}_d$ with index n is polynomial in n. It then follows easily from the Hadamard formula that

$$\psi(s) = \sum_{L\in\mathcal{L}_d} \frac{p_L(\alpha)}{[L]}\, s^{[L]}$$

has radius of convergence $\rho = \exp(-g(\alpha))$. Also, since the Taylor coefficients of $\psi(s)$ are nonnegative, we see that $\lim_{s\to\rho^-} \psi(s) = \infty$. It now follows that $\zeta_\alpha(s) = e^{\psi(s)}$ has radius of convergence $\rho = \exp(-g(\alpha))$. □

We are ready for the main result of this section.

Theorem 5.4 (Product Formula). Let α be a $\mathbb{Z}^d$-action for which $g(\alpha) < \infty$. Then

$$\zeta_\alpha(s) = \prod_{\gamma} \pi_d(s^{|\gamma|}), \tag{5.1}$$

where the product is taken over all finite orbits γ of α and this product converges for all $|s| < \exp(-g(\alpha))$.

Proof. Enumerate the finite orbits of α as $\gamma_1, \gamma_2, \dots$. Lemma 2.3 and Proposition 5.1 show that (5.1) is valid for the restriction of α to $\gamma_1 \cup \gamma_2 \cup \dots \gamma_n$ for all $n \geq 1$. Since everything in sight converges absolutely for $|s| < \exp(-g(\alpha))$, a standard argument shows that letting $n \to \infty$ proves (5.1) in general. □

Corollary 5.5. Let α be a $\mathbb{Z}^d$-action for which $\zeta_\alpha(s)$ has a positive radius of convergence. Then the Taylor series for $\zeta_\alpha(s)$ has integer coefficients.

Proof. Theorem 4.5 shows that the Taylor series of $\pi_d(s)$ has integer coefficients. The corollary now follows from the product formula. □

6. Algebraic examples

In this section we investigate the zeta function for some $\mathbb{Z}^d$-actions of algebraic origin. A general framework for studying such actions was first given by Kitchens and Schmidt [7]; see [15] for a systematic exposition.

Let $R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ be the ring of Laurent polynomials in d commuting variables. For $\mathbf{u} = (u_1, \dots, u_d)$ and $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ put $\mathbf{u}^{\mathbf{n}} = u_1^{n_1} \dots u_d^{n_d}$. A polynomial $f \in R_d$ then has the form

$$f(\mathbf{u}) = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) \mathbf{u}^{\mathbf{n}},$$

where $c_f(\mathbf{n}) \in \mathbb{Z}$ and all but finitely many of the $c_f(\mathbf{n})$ are zero. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the additive torus and $\mathbb{S} = \{e^{2\pi i t} : t \in \mathbb{T}\}$ be its multiplicative counterpart. For each $f \in R_d$ put

$$X_f = \Big\{ x \in \mathbb{T}^{\mathbb{Z}^d} : \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{k}+\mathbf{n}} = 0 \text{ for all } \mathbf{k} \in \mathbb{Z}^d \Big\}.$$

Then X_f is a compact group under coordinate-wise operations. The $\mathbb{Z}^d$ shift action α_f on X_f is defined by $(\alpha_f^{\mathbf{n}} x)_{\mathbf{k}} = x_{\mathbf{k}+\mathbf{n}}$. Each $\alpha_f^{\mathbf{n}}$ is clearly a continuous automorphism of X_f. In the framework of [7] the system (X_f, α_f) corresponds to the R_d-module $R_d/\langle f \rangle$.

Let us say that $f \in R_d$ is *expansive* if it has no zeros on $\mathbb{S}^d$. For example, $p(u, v) = 3 + u + v$ is expansive since $|3 + u + v| \geq 3 - |u| - |v| \geq 1$ provided $|u| = |v| = 1$. Similarly, $q(u, v) = 3 - u - v$ is also expansive. There is a natural notion of expansiveness for $\mathbb{Z}^d$-actions. A special case of work of Schmidt [14] is that α_f is expansive if and only if f is expansive.

We next turn to entropy (see Appendix A of [9] for definitions). For $0 \neq f \in R_d$ define the *Mahler measure* of f to be

$$M(f) = \exp\left(\int_{\mathbb{S}^d} \log |f| \right) = \exp\left(\int_0^\infty \cdots \int_0^\infty \log |f(e^{2\pi i t_1}, \dots, e^{2\pi i t_d})| \, dt_1 \dots dt_d \right).$$

By [9, Thm. 3.1], the topological entropy of α_f is given by $h(\alpha_f) = \log M(f)$. For instance, it is easy to verify that for $p(u, v) = 3 + u + v$ and $q(u, v) = 3 - u - v$ we have that $M(p) = M(q) = 3$, so that $h(\alpha_p) = h(\alpha_q) = \log 3$.

For $L \in \mathcal{L}_d$ put $\|L\| = \min\{\|\mathbf{n}\| : \mathbf{n} \in L \setminus \{0\}\}$. Then $\|L\| \to \infty$ means that "L goes to infinity in all directions." By [9, Thm. 7.1], if f is expansive then

$$h(\alpha_f) = \lim_{\|L\| \to \infty} \frac{1}{[L]} \log p_L(\alpha_f). \tag{6.1}$$

However, there are other ways for $[L] \to \infty$ without $\|L\| \to \infty$, and the growth rate for periodic points can be strictly larger for these. For instance, in Example 3.3 we have that

$$\lim_{n \to \infty} \frac{1}{n} \log p_{\mathbb{Z} \oplus n\mathbb{Z}}(\alpha) = \log k,$$

while it is easy to verify that

$$\lim_{\|L\| \to \infty} \frac{1}{[L]} \log p_L(\alpha) = h(\alpha) = 0.$$

For the $\mathbb{Z}^d$-actions α_f we will explicitly determine the growth rate of periodic points in terms of the Mahler measure of f on compact subgroups of $\mathbb{S}^d$.

Let $\mathcal{C}_d$ denote the collection of all compact subgroups of $\mathbb{S}^d$, and $\mathcal{K}_d \subset \mathcal{C}_d$ be the subset of infinite compact subgroups of $\mathbb{S}^d$. For $K \in \mathcal{C}_d$ define the *Mahler measure of f over K* to be

$$M_K(f) = \int_K \log |f|,$$

where the integral is with respect to Haar measure on K. By definition, $M(f) = M_{\mathbb{S}^d}(f)$.

Theorem 6.1. Let $f \in R_d$ be an expansive polynomial and α_f be the associated $\mathbb{Z}^d$-action. Then the growth rate of periodic points for α_f is given by

$$g(\alpha_f) = \sup_{K \in \mathcal{K}_d} \log M_K(f). \tag{6.2}$$

Furthermore, there is a $K_0 \in \mathcal{K}_d$ for which $g(\alpha_f) = \log M_{K_0}(f)$.

Before proving this result let us give two examples.

Example 6.2. (a) Let $d = 2$, $R_2 = \mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$, and $p(u, v) = 3 + u + v$. As noted above, $p(u, v)$ is expansive. In order to compute $g(\alpha_p)$, we first observe that if $f(\mathbf{u}) = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) \mathbf{u}^{\mathbf{n}} \in R_d$ has $|c_f(\mathbf{0})| = |f(\mathbf{0})| \geq \sum_{\mathbf{n} \neq 0} |c_f(\mathbf{n})|$, then Jensen's formula shows that $M(f) = |c_f(\mathbf{0})|$. Hence $M(p) = 3$, so that $h(\alpha_p) = \log 3$. Observe that

$$M_{\mathbb{S} \times \{1\}}(p) = M(4 + u) = 4 = M_{\{1\} \times \mathbb{S}}.$$

We show that the subgroups $\mathbb{S} \times \{1\}$ and $\{1\} \times \mathbb{S}$ give maximal Mahler measure for p over all subgroups in $\mathcal{K}_2$, so that by Theorem 6.1 we have that $g(\alpha_p) = \log 4$ and so $\zeta_{\alpha_p}(s)$ has radius of convergence $1/4$.

To see this, first note that any $K \in \mathcal{K}_2$ is either all of $\mathbb{S}^2$ or is 1-dimensional. In the latter case the connected component of the identity in K is the image of a rational line under the the exponential map $\mathbb{R}^2 \to \mathbb{S}^2$. Hence there is a vector $(a, b) \in \mathbb{Z}^2 \setminus \{0\}$ and a finite subgroup Ω of roots of unity in $\mathbb{S}^2$ such that

$$K = \bigcup_{t \in \mathbb{R}} \bigcup_{(\omega, \eta) \in \Omega} (\omega e^{2\pi i a t}, \eta e^{2\pi i b t}).$$

Hence

$$M_K(p) = \left(\prod_{(\omega, \eta) \in \Omega} M(3 + \omega u^a + \eta u^b) \right)^{1/|\Omega|}$$

Our observation above using Jensen's formula shows that

$$M(3+\omega u^a+\eta u^b)=\begin{cases}3 & \text{if } a\neq 0 \text{ and } b\neq 0,\\ |3+\omega| & \text{if } a=0 \text{ and } b\neq 0,\\ |3+\eta| & \text{if } a\neq 0 \text{ and } b=0.\end{cases} \tag{6.3}$$

Thus $M_K(p)$ is the geometric mean of numbers all of which are less that or equal to 4, so that $M_K(p)\le 4$ for all $K\in\mathcal{K}_2$. This proves that $g(\alpha_p)=\log 4$.

Hence $\zeta_{\alpha_p}(s)$ has radius of convergence 1/4. It is conjectured that it has the circle of radius 1/3 as natural boundary. Also, (6.3) shows that $\mathbb{S}\times\{1\}$ and $\{1\}\times\mathbb{S}$ are the only subgroups K in $\mathcal{K}_2$ for which $M_K(p)=4$.

(b) Again let $d=2$, but now use $q(u,v)=3-u-v$. As before, $q(u,v)$ is expansive and $M(q)=3$. We claim that $M_K(q)\le 3$ for all $K\in\mathcal{K}_2$. By (6.3), we need only consider subgroups of the form $\mathbb{S}\times\Omega_n$ or $\Omega_n\times\mathbb{S}$, where Ω_n is the group of nth roots of unity in $\mathbb{S}$. But for these,

$$M_{\mathbb{S}\times\Omega_n}(q)=M_{\Omega_n\times\mathbb{S}}(q)=\left(\prod_{k=0}^{n-1}M(3-e^{2\pi ik/n}-u)\right)^{1/n}$$

$$=\prod_{k=0}^{n-1}|3-e^{2\pi ik/n}|^{1/n}=(3^n-1)^{1/n}\le 3.$$

This proves that $g(\alpha_q)=\log 3$, so that $\zeta_{\alpha_q}(s)$ has radius of convergence 1/3. It is conjectured that it has the circle of radius 1/3 as natural boundary.

Proof of Theorem 6.1. Denote the annihilator of $L\in\mathcal{L}_d$ by $K_L=L^\perp\in\mathcal{C}_d$. By duality, $|K_L|=[L]$. Proposition 7.4 of [9] shows that

$$p_L(\alpha_f)=\prod_{\omega\in K_L}|f(\omega)|.$$

Hence

$$\frac{1}{[L]}\log p_L(\alpha_f)=\frac{1}{|K_L|}\sum_{\omega\in K_L}\log|f(\omega)|=\log M_{K_L}(f).$$

It is easy to see that $\mathcal{C}_d$ is compact with respect to the Hausdorff metric on compact subsets of $\mathbb{S}^d$. Since f is expansive, it follows that $\log|f|$ is continuous on $\mathbb{S}^d$, so that the function $K\mapsto\log M_K(f)$ is continuous on $\mathcal{C}_d$.

Suppose that $K\in\mathcal{K}_d$. Then there are lattices $L_n\in\mathcal{L}_d$ for which $K_{L_n}\to K$ in the Hausdorff metric. Since K is infinite, $[L_n]\to\infty$, so that

$$\log M_K(f)=\lim_{n\to\infty}\log M_{K_{L_n}}(f)=\lim_{n\to\infty}\frac{1}{[L_n]}\log p_{L_n}(\alpha_f)\le g(\alpha_f).$$

This proves that $\sup_{K\in\mathcal{K}_d}\log M_K(f)\le g(\alpha_f)$.

To prove the opposite inequality, choose $L_n\in\mathcal{K}_d$ with $[L_n]\to\infty$ and

$$\frac{1}{[L_n]}\log p_{L_n}(\alpha_f)\to g(\alpha_f).$$

Since $\mathcal{C}_d$ is compact, the subgroups $K_{L_n}\subseteq\mathbb{S}^d$ have a convergent subsequence $\{K_{L_{n_j}}\}$ converging to some $K_0\in\mathcal{C}_d$, where clearly $K_0\in\mathcal{K}_d$ since $[L_n]\to\infty$. Then

$$\log M_{K_0}(f)=\lim_{j\to\infty}\log M_{K_{L_{n_j}}}(f)=\lim_{n\to\infty}\frac{1}{[L_n]}\log p_{L_n}(\alpha_f)=g(\alpha_f).$$

This shows that $\sup_{K\in\mathcal{K}_d}\log M_K(f)=g(\alpha_f)$, and also that the supremum is attained at K_0. □

7. Questions and Problems

In the examples of $\mathbb{Z}^d$-actions α considered thus far, the zeta function $\zeta_\alpha(s)$ may have poles s with $|s|<\exp(-h(\alpha))$ but $\zeta_\alpha(s)$ has natural boundary $|s|=\exp(-h(\alpha))$. The quantity $h(\alpha)$ enters as the growth rate

$$\rho=\limsup_{\|L\|\to\infty}\frac{1}{[L]}\log p_L(\alpha) \tag{7.1}$$

as L goes to infinity in all directions. This leads to the main conjecture regarding the analytic behavior of $\zeta_\alpha(s)$.

Conjecture 7.1. Let α be a $\mathbb{Z}^d$-action and ρ be defined by (7.1). Then $\zeta_\alpha(s)$ is meromorphic in $|s|<e^{-\rho}$ and has the circle of radius $e^{-\rho}$ as natural boundary.

When $d=1$ the zeta function of a finitely determined system ϕ can be specified by a finite amount of data, namely the zeros and poles of the rational function $\zeta_\phi(s)$.

Problem 7.2. For "finitely determined" $\mathbb{Z}^d$-actions α such as shifts of finite type, is there a reasonable finite description of $\zeta_\alpha(s)$?

When $\alpha=\alpha_f$ is one of the algebraic examples described in §6, the numbers $M_K(f)$ for $K\in\mathcal{K}_d$ appear to play an essential role in describing $\zeta_\alpha(s)$. How much do these number tell about the polynomial $f\in R_d$?

Problem 7.3. Suppose that $f,g\in R_d$, and that $M_K(f)=M_K(g)$ for all $K\in\mathcal{K}_d$. What is the relationship between f and g? Must f and g differ by multiplicative factors that are monomials or generalized cyclotomic polynomials?

Problem 7.4. Compute explicitly the zeta function of the algebraic examples α_f discussed in §6.

Our zeta function $\mathbb{Z}^d$-actions can be generalized, as for a single transformation, to a "thermodynamic" setting by introducing a weight function $\theta\colon X\to(0,\infty)$, and defining

$$\zeta_{\alpha,\theta}(s)=\exp\left(\sum_{L\in\mathcal{L}_d}\left\{\sum_{x\in\operatorname{fix}_L(\alpha)}\prod_{\mathbf{k}\in\mathbb{Z}^d/L}\theta(\alpha^{\mathbf{k}}x)\right\}\frac{s^{[L]}}{[L]}\right),$$

where $\text{fix}_L(\alpha)$ is the set of points fixed by $\alpha^{\mathbf{n}}$ for all $\mathbf{n} \in L$. We obtain the usual zeta function ζ_α by using the weight function $\theta \equiv 1$. The following question was suggested to us by David Ruelle.

Problem 7.5. Compute explicitly the thermodynamic zeta function for the 2-dimensional Ising model, where α is the $\mathbb{Z}^2$ shift action on the space of configurations.

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THE DYNAMICAL THEORY OF TILINGS AND QUASICRYSTALLOGRAPHY

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ABSTRACT. A tiling x of R^n is *almost periodic* if a copy of any patch in x occurs within a fixed distance from an arbitrary location in x. Periodic tilings are almost periodic, but aperiodic almost periodic tilings also exist; for example, the well known Penrose tilings have this property. This paper develops a generalized symmetry theory for almost periodic tilings which reduces in the periodic case to the classical theory of symmetry types. This approach to classification is based on a *dynamical theory of tilings*, which can be viewed as a continuous and multidimensional generalization of symbolic dynamics.

1. INTRODUCTION

The purpose of this this paper is to describe a natural generalization of the standard theory of symmetry types for periodic tilings to a larger class of tilings called *almost periodic tilings*. In particular, a tiling x of R^n is called *almost periodic* [1] if a copy of any patch which occurs in x re-occurs within a bounded distance from an arbitrary location in x. Periodic tilings are clearly almost periodic since any patch occurs periodically, but there are also many aperiodic examples of almost periodic tilings—the most famous being the *Penrose tilings*, discovered in around 1974 by R. Penrose [18].

Ordinary symmetry theory is based on the notion of a symmetry group—the group of all rigid motions leaving an object invariant. The symmetry groups of periodic tilings are characterized by the fact that they contains a lattice of translations as a subgroup. In contrast, for aperiodic tilings the symmetry group contains no translations, and it is typically empty. Thus a generalization of symmetry theory to almost periodic tilings must be based on different considerations. In this paper we describe a generalization of symmetry theory that uses ideas from dynamical systems theory, applied 'tiling dynamical systems'. The simplest example of a tiling dynamical system consist of a translation invariant set of tilings, equipped with a compact metric

Partially supported by NSF DMS-9007831.

[1]Such tilings are also sometimes called *repetitive* or said to satisfy the *local isomorphism property*. The term *almost periodic* is used in the same sense as in topological dynamics.