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THE STRUCTURE OF SKEW PRODUCTS WITH ERGODIC GROUP AUTOMORPHISMS

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ABSTRACT
We prove that ergodic automorphisms of compact groups are Bernoulli shifts, and that skew products with such automorphisms are isomorphic to direct products. We give a simple geometric demonstration of Yuzvinskii's basic result in the calculation of entropy for group automorphisms, and show that the set of possible values for entropy is one of two alternatives, depending on the answer to an open problem in algebraic number theory. We also classify those algebraic factors of a group automorphism that are complemented.

1. Introduction

Over thirty years ago Halmos [11] noticed that continuous algebraic automorphisms of compact groups preserve Haar measure. Therefore they provide examples in ergodic theory which can be analysed in considerable detail because of the great amount of additional structure. For example, he showed that for compact abelian groups, ergodic automorphisms are also mixing. In 1964, Rohlin [31] strengthened this by showing that ergodic automorphisms of compact abelian groups are Kolmogorov automorphisms, that is, they obey a probabilistic zero-one law. Using ideas introduced by Ornstein together with some diophantine approximation arguments, in 1971 Katznelson [13] showed that ergodic automorphisms of finite dimensional tori are isomorphic to Bernoulli shifts, which is the strongest possible kind of mixing condition. A year later this result was extended to the infinite torus independently by the author [18], Chu [8], and Aoki and Totoki [2]. In fact, the algebraic techniques in [18] were strong enough to treat all ergodic automorphisms of compact abelian groups except those that we have dubbed "irreducible solenoidal automorphisms" in §5. One of our principle results, the Automorphism Theorem, is a proof that ergodic automorphisms of compact abelian groups are isomorphic to Bernoulli shifts. This result may be thought of as characterizing those measure-preserving transformations that can be given a compact abelian group automorphism structure.

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Skew product transformations were introduced by Anzai [1] to obtain certain counterexamples in ergodic theory. Such transformations, with a possibly nontrivial automorphism part, arise naturally in the study of automorphisms of nilmanifolds (see [29]). Our second main result, the Splitting Theorem, is that skew products with ergodic group automorphisms split into direct products, so that for ergodic automorphisms there is only one kind of skew product. This no longer holds for nonergodic automorphisms. The splitting can be obtained directly in certain cases by solving a related functional equation (e.g. the second proof of Theorem 3.1). Our approach here, however, is to prove both the Automorphism and Splitting Theorems at one stroke by showing that skew products with ergodic group automorphisms are Bernoulli modulo the base factor (see §2 for definitions). The latter result is called the Skew Product Theorem, and its proof makes essential use of the recent results of Thouvenot on relative ergodic theory.

There are some by-products and applications of our analysis. In §9 we give a simple geometric derivation of Yuzvinskii's formula for the entropy of an irreducible solenoidal automorphism, the central result in his computation [40] of the entropy of group automorphisms. We also show there that the set of possible values for the entropy of a group automorphism is either countable or all of $[0, \infty]$, depending on the answer to a problem in algebraic number theory that has been open for over forty years. In §8 we classify those algebraic factors of a group automorphism that split off with a Bernoulli complement. We make no use here of Rohlin's result that ergodic group automorphisms are Kolmogorov, so that along the way we have supplied an independent proof of this fact.

Recently and independently, G. Miles and R. K. Thomas have proved the Automorphism Theorem for general compact groups. In [21] they reduce the problem to the solenoid case. They then apply some recent and difficult results in simultaneous diophantine approximation due to Schmidt to establish some facts in [22], which, together with a Markov structure developed in their first paper, are used to dispose of the solenoid case in [23]. Our proofs are for abelian groups, but by using part of the known structure of general group automorphisms (see [21] or [41]), they extend easily to nonabelian groups. Roughly, the only ergodic automorphisms of nonabelian compact groups are shifts on a doubly infinite product of a nonabelian compact group, and these pose no problem. In particular, there are no ergodic automorphisms of a compact nonabelian Lie group [33]. Our proof of the solenoid case is somewhat shorter than [22] and [23], and avoids entirely any but the simplest number theory.

In his thesis, Marcuard [20] proves the Splitting Theorem for automorphisms of finite dimensional tori by using Thouvenot's relative ergodic theory combined with the observation that Katznelson's proof in [13] is in a sense translation invariant. In §4, as preparation for the irreducible solenoid case, we give an alternate proof of Marcuard's result which avoids his use of diophantine approximation. In another paper we will solve directly for toral automorphisms the related functional equation mentioned previously, which therefore gives a third proof of the Splitting Theorem for finite dimensional tori.

In §2 we sketch some necessary background material, including the relevant parts of Thouvenot's relative ergodic theory. Then in §3 we give a brief account of our results in the special case of a group shift, where the relative results are probably the clearest. As motivation for the difficult solenoid case, we present in §4 a new proof that ergodic automorphisms of finite dimensional tori are Bernoulli. In §5 we treat irreducible solenoidal automorphisms by using the ideas of the previous section together with a kind of independence on a totally disconnected subgroup. Totally disconnected groups are handled in §6, and in §7 these results are assembled to yield a proof for general groups. Applications mentioned above are in §8 and §9.

2. Preliminaries and statement of results

We collect here some terminology and basic results from ergodic theory, and formulate our basic results.

A *Lebesgue space* is a measure space consisting of a (possibly empty) continuous part that is isomorphic to the unit interval with Lebesgue measure, and a (possible empty) discrete part consisting of at most a countable number of atoms, together normalized to have measure 1. An invertible measure-preserving transformation between two Lebesgue spaces will be called simply a *map*. Two maps, U from X to itself and U' from X' to itself are isomorphic if there is a map $W: X \to X'$ such that $WU = U'W$.

If $\{\mathscr{A}_i : i \in I\}$ is a collection of σ-subalgebras of a Lebesgue space, then $\bigvee_{i \in I} \mathscr{A}_i$ denotes the smallest complete σ-subalgebra containing every $\mathscr{A}_i$. The collection $\{\mathscr{A}_i : i \in I\}$ is *independent* if any finite subcollection is independent in the usual probabilistic sense. If U is a map of the Lebesgue space $(X, \mathscr{N}, \nu)$ to itself, a σ-subalgebra $\mathscr{A}$ of $\mathscr{N}$ that is invariant under U is called a *factor* of U. A factor $\mathscr{A}$ of U is *Bernoulli* if there is a σ-subalgebra $\mathscr{B}$ of $\mathscr{A}$ such that $\{U^i\mathscr{B} : i \in \mathbf{Z}\}$ is independent and $\bigvee_{-\infty}^{\infty} U^i\mathscr{B} = \mathscr{A}$. The map U itself is a *Bernoulli map* if $\mathscr{N}$ is a Bernoulli factor.

We shall need relative versions of Ornstein's Factor and Monotone Theorems for Bernoulli factors ([25] and [24]). These versions were originally established by Thouvenot [36] for finite entropy factors only. The modifications necessary to extend his work to factors if infinite entropy are given in the Appendix.

Let $\mathscr{H}$ be a factor of a map U. A factor $\mathscr{A}$ is *Bernoulli* mod $\mathscr{H}$ if there is a Bernoulli factor $\mathscr{B}$ that is independent of $\mathscr{H}$ and such that $\mathscr{A} \vee \mathscr{H} = \mathscr{B} \vee \mathscr{H}$. U is a *Bernoulli map* mod $\mathscr{H}$ if $\mathscr{N}$ is Bernoulli mod $\mathscr{H}$. Letting $\mathscr{H}$ be the trivial σ-subalgebra corresponds to the absolute case.

RELATIVE FACTOR THEOREM. *Factors of Bernoulli maps* mod $\mathscr{H}$ *are Bernoulli* mod $\mathscr{H}$.

RELATIVE MONOTONE THEOREM. *If* $\{\mathscr{A}_i : i = 1, 2, \cdots\}$ *is an increasing sequence of Bernoulli factors* mod $\mathscr{H}$, *then* $\vee_1^\infty \mathscr{A}_i$ *is also a Bernoulli factor* mod $\mathscr{H}$.

We remark that it follows from the definition that if $\mathscr{A}_1 \subset \mathscr{A}_2 \subset \mathscr{A}_3$, and $\mathscr{A}_2$ is Bernoulli mod $\mathscr{A}_1$, $\mathscr{A}_3$ is Bernoulli mod $\mathscr{A}_2$, then $\mathscr{A}_3$ is Bernoulli mod $\mathscr{A}_1$.

Finite, ordered, measurable partitions of a Lebesgue space will be denoted by Greek letters $\alpha, \beta, \cdots$, with possible embellishments. If $\{\alpha_i : m \leqq i \leqq n\}$ is a finite sequence of partitions, then $\vee_m^n \alpha_i$ denotes the common refinement of these partitions ordered lexicographically. Identifying a partition with the σ-algebra it generates, the symbol $\vee_{i \in I} \alpha_i$ over an infinite index set I is interpreted to be the smallest complete σ-subalgebra containing every α_i. The entropy of α is denoted by $h(\alpha)$, and if U is a map, the entropy of U on α is denoted by $h(U, \alpha)$. The entropy of U is then $h(U) = \sup_\alpha h(U, \alpha)$, where the supremum is taken over all finite partitions α.

Let G be a separable compact group, which we will write additively. Then G together with the completion $\mathscr{M}$ of the Borel σ-algebra and Haar measure μ forms a Lebesgue space. A continuous algebraic automorphism of G, which we hereafter call simply an automorphism, preserves μ. As mentioned in §1, our first main result is the following.

AUTOMORPHISM THEOREM. *An ergodic automorphism of a compact abelian group is isomorphic to a Bernoulli map.*

If $(X, \mathscr{N}, \nu)$ is a Lebesgue space, then $X \times G$ together with product measure $\nu \otimes \mu$ on the completed product σ-algebra $\mathscr{N} \otimes \mathscr{M}$ is again a Lebesgue space. Let U be an ergodic map of X, S an automorphism of G, and $\phi : X \to G$ measurable. Define a map $V = U \times_\phi S$ by $V(x, g) = (Ux, Sg + \phi(x))$. V is called the *skew product* of U with S with skewing function ϕ. If $\phi(x)$ is always the identity of G, then $U \times_\phi S$ is just the direct product $U \times S$. The σ-algebra $\mathscr{N}$ is identified with $\mathcal{N} \times G$ in $\mathcal{N} \otimes \mathcal{M}$, and is called the *base factor* of V. The *addition formula* ([41], [34]) for the entropy of a skew product is $h(U \times_\phi S) = h(U) + h(S)$.

A skew product $U \times_\phi S$ *splits* if there is an isomorphism of it to the direct product $U \times S$ which is the identity on the base factor $\mathcal{N}$, i.e. a map W from $X \times G$ to itself such that $W(U \times_\phi S) = (U \times S)W$, and $W(x, g) = (x, W_x g)$, where W_x is a map of G for every $x \in X$. Our second main result is formulated as follows.

SPLITTING THEOREM. *Skew products with ergodic automorphisms of compact abelian groups split.*

We will prove here both the Automorphism and Splitting Theorems at one blow by showing the following.

SKEW PRODUCT THEOREM. *Skew products with ergodic automorphisms of compact abelian groups are Bernoulli* mod *the base factor.*

The Automorphism Theorem follows by taking X to have one point. To obtain the Splitting Theorem, observe that if $U \times_\phi S$ is Bernoulli mod $\mathcal{N}$, then it is isomorphic to the direct product of U with a Bernoulli map S' which has, by the addition formula, the same entropy as S. Since S is also Bernoulli, S is isomorphic to S'. Then $U \times_\phi S \cong U \times S' \cong U \times S$, where both isomorphisms preserve the base factor.

If H is a closed subgroup of G that is invariant under S, we denote by S_H the restriction of S to H, and by $S_{G/H}$ the factor automorphism of S on G/H. If $\mathcal{M}(H)$ is the σ-subalgebra of those sets in $\mathcal{M}$ that are unions of cosets of H, then $\mathcal{M}(H)$ is a translation invariant factor of S. Hence $\mathcal{N} \otimes \mathcal{M}(H)$ is a factor of $V = U \times_\phi S$ and can be identified with the skew product $V/H = U \times_{\bar{\phi}} S_{G/H}$, where $\bar{\phi}$ is the image of ϕ under the quotient map $\pi: G \to G/H$.

We can regard V as a skew product of V/H with S_H in the following way. Let $\psi: G/H \to G$ be a Borel cross section to π (see [5; IX, 6.8]), so that $\pi\psi$ is the identity on G/H. Identify $X \times G$ with $X \times (G/H) \times H$ via $(x, g) \leftrightarrow (x, \pi g, g - \psi\pi g)$. Then it is simple to check that

$$\begin{aligned} V(x, g) &= ((U \times_{\bar{\phi}} S_{G/H})(x, \pi g),\ S_H(g - \psi\pi g) + \theta(x, \pi g)) \\ &= (V/H \times_\theta S_H)(x, \pi g, g - \psi\pi g), \end{aligned}$$

where

$$\theta(x, \pi g) = \phi(x) + S\psi\pi g - \psi\pi(Sg + \phi(x)).$$

Thus V is the skew product of V/H with S_H with skewing function θ.

3. Group shifts

There is one kind of ergodic group automorphism, a group shift, which is clearly a Bernoulli map, and for which the proof of the Splitting and Skew Product Theorems is particularly simple. We treat this case both as motivation for what follows, and because the results are used when dealing with totally disconnected and nonabelian groups. It is here that an idea suggested by B. Weiss in §7(c) of his survey [38] meets with complete success.

Let $\{G_i\}$ be a doubly infinite sequence of copies of some separable compact group G_0. The product group $G = \prod_{-\infty}^{\infty} G_i$ is again separable. The *group shift on* G_0 is the shift automorphism S of G defined by $S(\cdots, g_{-1}, g_0, g_1, \cdots) = (\cdots, g_0, g_1, g_2, \cdots)$. Let $\pi_0\colon G \to G_0$ be the projection onto the 0th coordinate. The σ-subalgebra $\mathcal{B} = \pi_0^{-1}(\mathcal{M}_0)$ has $\{S^i\mathcal{B} : i \in \mathbf{Z}\}$ independent since Haar measure on G is product measure, and $\bigvee_{-\infty}^{\infty} S^i\mathcal{B} = \mathcal{M}$ because the cylinder sets of $\mathcal{M}$ are in this span. Then S is a Bernoulli map with independent generator $\mathcal{B}$.

THEOREM 3.1. *The Splitting and Skew Product Theorems hold for group shifts.*

PROOF. We give two proofs. The first uses the observation that the time zero σ-subalgebra of a group shift moves independently under the shift in a translation invariant way to yield a Bernoulli complement to the base factor. This proves the Skew Product Theorem, from which the Splitting Theorem follows. This kind of translation invariant independence is a simple form of the relative very weak Bernoulli criterion we will establish later for other groups. The second proof splits the skew product directly, producing the required isomorphism by solving a related functional equation. Since group shifts are Bernoulli, the Splitting Theorem immediately implies the Skew Product Theorem.

To begin the first proof, let U be an ergodic map of $(X, \mathcal{N}, \nu)$, S be the group shift on G_0, and $V = U \times_\phi S$. Let $\mathcal{B}_0 = \pi_0^{-1}(\mathcal{M}_0)$ be the time zero σ-subalgebra for S, and π_X and π_G be the coordinate projections of $X \times G$ onto X and G. We will show that if $\mathcal{B} = \pi_G^{-1}(\mathcal{B}_0)$, then $\{V^i\mathcal{B} : i \in \mathbf{Z}\}$ are independent and together with $\mathcal{N}$ generate $\mathcal{N} \otimes \mathcal{M}$, which will prove the Skew Product Theorem.

First note that iterates of V have the form $V^i(x, g) = (U^i x, S^i g + \psi_i(x))$, $i \in \mathbf{Z}$, where $\psi_1(x) = \phi(x)$, $\psi_2(x) = S\phi(x) + \phi(Ux)$, and so on. An atom in $\bigvee_{-n}^{n} V^i\mathcal{B}$ has the form

$$A = \bigcap_{-n}^{n} V^i \pi_G^{-1}(E_i), \qquad E_i \in \mathcal{B}_0.$$

Then, since Haar measure is translation invariant and on G is product measure, we have for every x that

$$\begin{aligned}\mu(A \cap \pi_X^{-1}(x)) &= \mu\left[\bigcap_{-n}^{n} (S^i E_i + \psi_i(x))\right] \\ &= \prod_{-n}^{n} \mu_i(S^i E_i + \psi_i(x)) \\ &= \prod_{-n}^{n} \mu_0(E_i) \\ &= \prod_{-n}^{n} (\nu \otimes \mu)(\pi_G^{-1} E_i).\end{aligned}$$

Since the answer is independent of x, we simultaneously conclude that $\{V^i\mathcal{B} : i \in \mathbf{Z}\}$ is independent and has span independent of $\mathcal{N}$.

To prove that $\mathcal{B}$ together with $\mathcal{N}$ generate $\mathcal{N} \otimes \mathcal{M}$, note that $S^i\mathcal{B} + g = S^i\mathcal{B}$ for any $g \in G$. Hence for every x we have

$$\pi_G\left(\bigvee_{-\infty}^{\infty} V^i\mathcal{B} \mid \pi_X^{-1}(x)\right) = \bigvee_{-\infty}^{\infty} (S^i\mathcal{B} + \psi_i(x)) = \mathcal{M},$$

and since $\mathcal{N}$ separates the points of X, $\mathcal{N} \vee \bigvee_{-\infty}^{\infty} V^i\mathcal{B} = \mathcal{N} \otimes \mathcal{M}$.

For the second proof, due to Miles and Thomas [21], we will produce an isomorphism W of $U \times_\phi S$ with $U \times S$ of the form $W(x, g) = (x, g + f(x))$, where $f \colon X \to G$ is measurable. The relation $W(U \times_\phi S) = (U \times S)W$ is equivalent to the functional equation

$$\phi(x) = Sf(x) - f(Ux), \tag{3.1}$$

where ϕ, S, and U are known and it required to find a measurable solution $f \colon X \to G$. We shall solve (3.1) when S is a group shift.

Since $\phi(x) \in G = \Pi_{-\infty}^{\infty} G_i$, $\phi(x) = (\cdots, \phi_{-1}(x), \phi_0(x), \phi_1(x), \cdots)$. Writing $f(x) = (\cdots, f_{-1}(x), f_0(x), f_1(x), \cdots)$, then (3.1) becomes

$$\phi_i(x) = f_{i+1}(x) - f_i(Ux). \tag{3.2}$$

We simply put $f_0(x) \equiv 0$ in G_0, and use (3.2) to find the other f_j inductively. Thus $f_1(x) = \phi_0(x)$, $f_2(x) = \phi_0(Ux) + \phi_1(x)$, and in general

$$f_k(x) = \begin{cases} \sum_{j=0}^{k-1} \phi_j(U^{k-j-1}x) & (k \geqq 1) \\ \sum_{j=k}^{-1} \phi_j(U^{k-j-1}x) & (k \leqq -1) \end{cases}$$

We have therefore produced an explicit isomorphism between $U \times_\phi S$ and $U \times S$ which preserves the base factor. Such an isomorphism is not unique, for we could have started with an arbitrary $f_0(x)$ in the above. Also, by solving the functional equation (3.1) directly, we have produced a factor complementary to $\mathcal{N}$ which is invariant under the natural action of G on $X \times G$. Indeed, the existence of such a factor is equivalent to solving (3.1). Our proof for general group automorphisms of the existence of Bernoulli complements in the Skew Product Theorem uses relative ergodic theory, and does not allow us to conclude that they are invariant under G.

Our calculation in the first proof that shows independence of $\{V^i\mathcal{B}\}$ is essentially the observation that since $\{S^i\mathcal{B}_0\}$ is independent and each term is translation invariant, for any sequence of elements $g_i \in G$ we have $\{S^i\mathcal{B}_0 + g_i\}$ is independent. A weaker form of such translation invariant independence is a key element later on. Unfortunately, this stronger situation occurs only for group shifts.

THEOREM 3.2. *Suppose that S is an automorphism of a compact abelian group G, and that $\mathcal{B}$ is a translation invariant σ-subalgebra of $\mathcal{M}$ such that $\{S^i\mathcal{B} : i \in \mathbf{Z}\}$ is independent and generates $\mathcal{M}$. Then there is an algebraic isomorphism of S with a group shift that carries $\mathcal{B}$ to the time zero σ-subalgebra.*

PROOF. Since $\mathcal{B}$ is translation invariant, there is a closed subgroup H of G such that $\mathcal{B} = \mathcal{M}(H)$ (see [17]). Define $\rho : G \to \Pi_{-\infty}^{\infty} G/H$ by $\rho(g) = \{S^i g + H : i \in \mathbf{Z}\}$. Then ρ is clearly a continuous algebraic homomorphism compatible with S and the group shift on G/H. Since $\bigvee_{-\infty}^{\infty} S^i\mathcal{B} = \mathcal{M}$, it follows that $\bigcap_{-\infty}^{\infty} S^i H$ is trivial, which implies that ρ is injective. To show that ρ is surjective, let $\{K_n\}$ be a sequence of compact subsets of G with positive Haar measure that decrease to the identity. For any elements $g_i \in G$, since $g_i + K_n + H \in \mathcal{B}$, the independence of $\{S^i\mathcal{B}\}$ shows that for each n and m

$$\mu\left[\bigcap_{-m}^{m} S^{-i}(g_i + K_n + H)\right] = \prod_{-m}^{m} \mu(S^{-i}(g_i + K_n + H)) > 0.$$

Let $n \to \infty$, and noting that all sets involved are compact and decreasing in n, we have that for every m,

$$\bigcap_{-m}^{m} S^{-i}(g_i + H) \neq \emptyset.$$

Again using compactness, letting $m \to \infty$ yields

$$\bigcap_{-\infty}^{\infty} S^{-i}(g_i + H) \neq \emptyset.$$

Any element g in this intersection has $\rho(g) = \{g_i + H\}$, which proves that ρ is surjective. We have defined H so that $\mathcal{B} = \mathcal{M}(H)$, and hence $\rho(\mathcal{B})$ is the time zero σ-subalgebra of the group shift on G/H.

4. Finite dimensional tori

We prove here our basic results for automorphisms of finite dimensional tori. The major stumbling block to extending the geometric ideas of Ornstein and Weiss [39] from tori of dimension two to those of higher dimensions has been handling those automorphisms in whose Jordan form the blocks corresponding to eigenvalues of modulus one contain off-diagonal ones. This difficulty is overcome in previous proofs ([13], [23]) by invoking some diophantine approximation arguments. However, here we decompose the automorphism into a succession of skew products with automorphisms whose characteristic polynomial is irreducible. We then prove the Skew Product Theorem for such automorphisms by using a translation invariant version of the Ornstein–Weiss technique to establish a criterion which Thouvenot proved was sufficient for relative Bernoullicity.

Strictly speaking, this proof is a special case of that in the next section. We have given it separately here both because it provides a new proof of the Automorphism Theorem for finite dimensional tori, and also because certain geometric ideas, which may be lost in the more complicated situation later, stand out clearest for tori.

Let us first introduce some notation and terminology. $\mathbf{T}^d$ denotes the d-dimensional torus $\mathbf{R}^d/\mathbf{Z}^d$, whose dual group is $\mathbf{Z}^d$ under the usual pairing. Maps will act on the left, so that elements of $\mathbf{R}^d$, $\mathbf{T}^d$, and $\mathbf{Z}^d$ are columns, even though we will usually write then as rows for typographical simplicity.

An automorphism S of $\mathbf{T}^d$ comes from a linear isomorphism $\tilde{S}$ of $\mathbf{R}^d$ for which $\tilde{S}\mathbf{Z}^d = \mathbf{Z}^d$. This amounts to requiring that the matrix of S have integral entries and determinant ± 1. The matrix of the dual automorphism T of $\mathbf{Z}^d$ is then the transpose of the matrix of S.

It is easy to show that an automorphism S of a compact abelian group is ergodic if and only if its dual automorphism T is aperiodic, that is, the only element of the dual group periodic under T is the identity. For toral automorphisms, this is equivalent to requiring that the matrix of S have no roots of unity as eigenvalues.

In order to apply some results from linear algebra, we embed the dual group $\mathbf{Z}^d$ into the d-dimensional rationals $\mathbf{Q}^d$ in the obvious way. The automorphism T extends to a rational vector space isomorphism of $\mathbf{Q}^d$ that we also call T. A coordinate-free approach, used in the next section, replaces $\mathbf{Q}^d$ with the tensor product of $\mathbf{Z}^d$ with $\mathbf{Q}$, but our ad hoc embedding, used also with the infinite torus in [18], is more direct and explicit.

As a linear map of $\mathbf{Q}^d$, T has characteristic polynomial $q(x)$ which is monic, with integral coefficients, and constant term ± 1. We will abbreviate these conditions on a polynomial $q(x)$ by saying that $q(x)$ is *unimodular*. Let the irreducible factorization of $q(x)$ in $\mathbf{Q}[x]$ be $q(x) = \Pi_i q_i(x)^{e_i}$. By Gauss's Lemma, we can assume that each $q_i(x)$ is monic with integral coefficients, and it follows that each is unimodular since their product is.

The primary decomposition for linear operators [12, p. 180] says that if V_i is the kernel of $q_i(T)^{e_i}$, then $\mathbf{Q}^d = \bigoplus_i V_i$. This comes from considering $\mathbf{Q}^d$ as a $\mathbf{Q}[x]$-module, where $\mathbf{Q}[x]$ is a principle ideal domain, and applying the structure theorem for modules over such domains. If we now apply the rational canonical form [12, p. 196], each V_i splits into a direct sum of cyclic subspaces for T, where on each cyclic subspace the minimal polynomial, which coincides with characteristic polynomial, must be a power of $q_i(x)$. Thus we can write $\mathbf{Q}^d = \bigoplus_j W_j$, where each W_j is a T-invariant cyclic subspace on which the minimal polynomial of T is the power of some irreducible polynomial $p_j(x)$ (each $p_j(x)$ is one of the $q_i(x)$ appearing in the factorization of $q(x)$).

Let w_j be a cyclic vector for W_j under T. Since any power of $p_j(x)$ is unimodular, the subgroup Λ_j of W_j generated by the powers of T on w_j is just the group generated by $\{T^k w_j: 0 \leqq k \leqq \dim W_j - 1\}$; that is, Λ_j is a lattice in W_j whose rank is $\dim W_j$. By replacing each w_j with an appropriate rational multiple if necessary, we can assume that $\mathbf{Z}^d \subset \bigoplus_j \Lambda_j = \Lambda$. Since $\mathbf{Z}^d$ has full rank in $\mathbf{Q}^d$, the quotient $\Lambda/\mathbf{Z}^d$ is finite.

Let G be the dual of Λ. The dual of the inclusion of $\mathbf{Z}^d \subset \Lambda$ is a quotient homomorphism $\pi: G \to \mathbf{T}^d$ with finite kernel. The automorphism T_Λ of Λ has dual an automorphism $\bar{S}$ of G which commutes with π. If G_j is the dual of Λ_j, then $G = \Pi_j G_j$, and $\bar{S}$ on the factor G_j has dual automorphism T_{Λ_j} which has a cyclic vector with minimal polynomial the power of an irreducible polynomial.

We claim that to prove the Skew Product Theorem for S, it suffices to prove it for $\bar{S}$. For if $U \times_\phi S$ is a skew product, let $\bar{\phi}: X \to G$ be any measurable function with $\pi\bar{\phi} = \phi$. Then $U \times_\phi S$ is isomorphic to the factor $\mathcal{N} \otimes \mathcal{M}(\ker \pi)$ of $U \times_{\bar{\phi}} \bar{S}$, and by the Relative Factor Theorem, knowing the Skew Product Theorem for $\bar{S}$ would give it also for S.

Also, from §2 we have that a skew product with $\bar{S}$ is a finite sequence of skew products with $\bar{S}_{G_j}$.

Thus we are reduced to the case when (simplifying notation) the dual automorphism T of S has cyclic vector w which generates all of the dual group Λ, such that the minimal polynomial of T is the power $p(x)^e$ of an irreducible $p(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0$. With respect to the basis $\{w, Tw, \cdots, T^{d-1}w,$

$p(T)w, p(T)Tw, \cdots, p(T)T^{d-1}w, \cdots, p(T)^{e-1}w, P(T)^{e-1}Tw, \cdots, P(T)^{e-1}T^{d-1}w\}$ the matrix of T is

$$\begin{bmatrix} \boxed{C(p)} & & & & \\ 1 & \boxed{C(p)} & & & \\ & 1 & \cdot & & \\ & & \cdot & \cdot & \\ & & & 1 & \boxed{C(p)} \end{bmatrix}$$

where there are e copies of

$$C(p) = \begin{bmatrix} 0 & & & & & -a_0 \\ 1 & 0 & & & & -a_1 \\ & 1 & 0 & & & -a_2 \\ & & \cdot & & & \cdot \\ & & & \cdot & & \cdot \\ & & & & \cdot & \cdot \\ & & & & 1 & -a_{d-1} \end{bmatrix}$$

the companion matrix to $p(x)$.

With respect to this basis, the dual of Λ splits as $\mathbf{T}_1^d \times \cdots \times \mathbf{T}_e^d$. Because $p(T)^j\Lambda$ is T-invariant, T is well defined on the quotient $\Lambda/p(T)^j\Lambda$, whose dual $\mathbf{T}_1^d \times \cdots \times \mathbf{T}_j^d$ is therefore invariant under S $(1 \leqq j \leqq e)$. If $C(p)'$ is the transpose of $C(p)$, then matrix of S on $\mathbf{T}_1^d \times \mathbf{T}_2^d$ is

$$\begin{bmatrix} \boxed{C(p)'} & 1 \\ & \boxed{C(p)'} \end{bmatrix}$$

and is a skew product of $C(p)'$ with $C(p)'$. Explicitly, if $(t^{(1)}, t^{(2)}) \in \mathbf{T}_1^d \times \mathbf{T}_2^d$, then $S(t^{(1)}, t^{(2)}) = (St^{(1)}, St^{(2)} + t_d^{(1)})$. Continuing in this fashion, we see that a skew product with S is built up as a finite sequence of skew products with $C(p)'$.

Thus the Skew Product Theorem, and consequently the Automorphism and Splitting Theorems, are implied by the following.

THEOREM 4.1. *The Skew Product Theorem holds for ergodic toral automorphisms whose matrix is the transpose of the companion matrix of an irreducible polynomial.*

Before launching into the proof, we first describe the relative version of very weak Bernoullicity due to Thouvenot, and prove some preliminary results needed to implement this criterion.

We begin with some notation. If $\alpha = \{A_1, A_2, \cdots\}$ is a partition of (X, ν), the *distribution of* α is the vector $d(\alpha) = (\nu(A_1), \nu(A_2), \cdots)$. If $\nu(E) > 0$, the restriction of α to E is $\alpha \mid E = \{A_1 \cap E, A_2 \cap E, \cdots\}$. A property is said to hold for ε*-almost every atom* of α if the union of the atoms of α for which it holds has measure greater than $1 - \varepsilon$. The *partition distance* between two partitions $\alpha = \{A_1, A_2, \cdots\}$ and $\beta = \{B_1, B_2, \cdots\}$ of the same space is $d(\alpha, \beta) = \Sigma_i \nu(A_i \Delta B_i)$. If $\{\alpha_i\}_1^n$ and $\{\beta_i\}_1^n$ are sequences of partitions of (X, ν_X) and (Y, ν_Y), respectively, then define

$$\bar{d}[\{\alpha_i\}_1^n, \{\beta_i\}_1^n] = \inf_{\psi} \frac{1}{n} \sum_{i=1}^{n} d(\psi\alpha_i, \beta_i),$$

where the infimum is taken over all maps $\psi: X \to Y$. A sequence of partitions $\{\alpha_i : i \in \mathbf{Z}\}$ of a measure space is called *very weak Bernoulli* if for every $\varepsilon > 0$ there is a $k > 0$ and an $n > k/\varepsilon$ such that for all $m > 0$, we have for ε-almost every atom $A \in \bigvee_{-m-k}^{-k} \alpha_i$ that

$$\bar{d}[\{\alpha_i \mid A\}_1^n, \{\alpha_i\}_1^n] < \varepsilon.$$

If V is a map, then a partition α is called *very weak Bernoulli for* V if the sequence $\{V^{-i}\alpha : i \in \mathbf{Z}\}$ is very weak Bernoulli.

The importance of this notion is that it gives a condition that can be checked in specific cases, and that if α is very weak Bernoulli for V, then $\bigvee_{-\infty}^{\infty} V^{-i}\alpha$ is a Bernoulli factor of V.

Thouvenot's relative version of very weak Bernoulli [37] in the context of skew products takes the following form, which is easily shown equivalent to his original formulation.

Let U be an ergodic map of $(X, \mathcal{N}, \nu)$, S be an automorphism of $(\mathbf{T}^d, \mathcal{M}, \mu)$, and $V = U \times_\phi S$ be a skew product of U with S. Denote the projections of $X \times \mathbf{T}^d$ onto X and $\mathbf{T}^d$ by π_X and π_d, respectively. Recall that we identify $\mathcal{N}$ with $\pi_X^{-1}(\mathcal{N})$, so that $\mathcal{N}$ is a factor of V. For $x \in X$, the fiber $\pi_X^{-1}(x) = \{x\} \times \mathbf{T}^d$ carries a natural measure structure, and if α is a partition of $X \times \mathbf{T}^d$, then $\alpha \mid \pi_X^{-1}(x)$ is defined for almost every x. Such a partition α is called *very weak Bernoulli* mod $\mathcal{N}$ *for* V if the sequence $\{V^{-i}\alpha \mid \pi_X^{-1}(x)\}$ is very weak Bernoulli for almost every $x^\dagger$. By relativising the absolute proof, Thouvenot [36, lemme 6] shows that if α is very weak Bernoulli mod $\mathcal{N}$ for V, then $\bigvee_{-\infty}^{\infty} V^{-i}\alpha$ is Bernoulli mod $\mathcal{N}$.

† The lack of stationarity of this sequence is the reason for our introducing above the notion of a very weak Bernoulli sequence of partitions.

A partition α of $\mathbf{T}^d$ is *smooth* if the atoms of α have piecewise smooth boundaries. We will show that the relative very weak Bernoulli criterion holds for partitions of the form $\pi_d^{-1}(\alpha)$, where α is a smooth partition of $\mathbf{T}^d$. This will prove Theorem 4.1, for by taking an increasing sequence $\{\alpha_j\}$ of smooth partitions of $\mathbf{T}^d$ for which $\bigvee_1^\infty \alpha_j = \mathcal{M}$, we will have an increasing sequence of factors $\mathcal{A}_j$ of V that are Bernoulli mod $\mathcal{N}$, and such that $\mathcal{N} \vee \mathcal{A}_j$ increases to $\mathcal{N} \otimes \mathcal{M}$. The Relative Monotone Theorem then implies Theorem 4.1. If S is hyperbolic, it is easy to produce a smooth generator, and use of the Relative Monotone Theorem could be avoided. However, for nonhyperbolic automorphisms it seems more difficult to produce smooth generators.

Let $V = U \times_\phi S$, and recall that the functions $\psi_i: X \to \mathbf{T}^d$ were defined in §3 by $V^i(x,t) = (U^i x, S^i t + \psi_i(x))$. If α is a smooth partition of $\mathbf{T}^d$, then restricted to the fiber $\pi_X^{-1}(x)$ the sequence of partitions $\{V^{-i}\pi_d^{-1}(\alpha)\}$ is $\{S^{-i}\alpha - S^{-i}\psi_i(x)\}$, as a straightforward computation shows. We can therefore complete the proof by showing that for any sequence $\{t_i\}$ in $\mathbf{T}^d$, $\{S^{-i}\alpha + t_i\}$ is very weak Bernoulli.

To prove the very weak Bernoulli criterion, we use the following elementary result of Ornstein and Weiss [39, lemma 1.3]. If $\{\alpha_i\}_1^n$ is a sequence of partitions of (X, ν_X), where $\alpha_i = \{A_{i1}, A_{i2}, \cdots\}$, define the $\{\alpha_i\}_1^n$-*name* of $x \in X$ to be $\{a_i(x)\}_1^n$, where $x \in A_{ia_i(x)}$. Let $\{\beta_i\}_1^n$ be a similar sequence on (Y, ν_Y), with $\{\beta_i\}_1^n$-names $\{b_i(y)\}_1^n$. An ε-*map* $\theta: X \to Y$ is a transformation such that there is a set $X_0 \subset X$ with $\nu_X(X_0) > 1 - \varepsilon$, and for any set $E \subset X_0$ we have

$$|\nu_Y(\theta E)/\nu_X(E) - 1| < \varepsilon.$$

Let $\delta(0) = 1$ and $\delta(k) = 0$ for $k \neq 0$.

LEMMA 4.2. *With the notation above, if there is an ε-map $\theta: X \to Y$ and a set $E \subset X$ with $\nu_X(E) < \varepsilon$ and*

$$\frac{1}{n}\sum_{i=1}^n \delta(a_i(x) - b_i(\theta x)) < \varepsilon \qquad (x \in X \backslash E),$$

then

$$\bar{d}[\{\alpha_i\}_1^n, \{\beta_i\}_1^n] < 16\varepsilon.$$

Our application of this result to prove Theorem 4.1 is a translation invariant analogue of the Ornstein–Weiss technique, except that we prove certain uniform distribution statements directly rather than appeal to Rohlin's result, as is done in [39]. Given a smooth partition α of $\mathbf{T}^d$ and $\varepsilon > 0$, we will introduce an auxiliary partition $\beta = \{C_j\}$ of $\mathbf{T}^d$ into "mapping boxes". There will exist a k depending on α and ε such that for every $m > 0$, most atoms of $\bigvee_{-m-k}^{-k}(S^{-i}\alpha + t_i)$ will be nearly uniformly distributed in each of the C_j, and will consist mostly of "sheets" in the unstable direction. This will allow us to define an ε-map from most atoms A to $\mathbf{T}^d$ by defining it "locally" from each $A \cap C_j$ to C_j. This local mapping has the property that a point and its image are close and their difference is in the stable direction. Hence the forward iterates of a point and its image remain close forever, and smoothness of α guarantees that their $\{S^{-i}\alpha\}$-names agree most of the time. An application of Lemma 4.2 then proves the required $\bar{d}$ closeness for very weak Bernoulli.

We shall now describe some of the geometry of toral automorphisms. The higher dimensional case differs in two respects from the two-dimensional case. The first is that the eigenvalues of ergodic automorphisms of $\mathbf{T}^2$ must be real, so that the eigenspaces are always one-dimensional. In general, the presence of complex eigenvalues and hence two-dimensional eigenspaces means that slightly more care is needed when using Weyl's Theorem to obtain uniform distribution. The second is the possible presence of eigenvalues of modulus 1. Since we are assuming that the characteristic polynomial is irreducible, these eigenvalues are nonrepeated, and make no difference in the proof. If, however, we tried to apply this proof to the repeated eigenvalue case, then polynomial growth in the stable direction caused by off-diagonal ones in the Jordan form would necessitate some delicate control over the rate of uniform distribution in Weyl's Theorem. This is in fact the approach of Miles and Thomas ([22], [23]).

We will assume for the remainder of this section that S is an ergodic automorphism of $\mathbf{T}^d$ whose matrix is the transpose of the companion matrix of an irreducible polynomial. If Φ: $\mathbf{R}^d \to \mathbf{T}^d$ is the natural quotient map, then there is, as mentioned before, a linear isomorphism $\tilde{S}$ of $\mathbf{R}^d$ such that $\Phi\tilde{S} = S\Phi$, and whose matrix is that of S. For each eigenvalue λ of $\tilde{S}$, let W_λ be the eigenspace corresponding to λ and $\bar{\lambda}$. Then dim W_λ is 1 or 2 depending on whether λ is real or not. In order to avoid repetitions, we make the convention that sums and products indexed by the eigenvalues λ are over only those λ whose imaginary part is nonnegative.

There is a metric on each W_λ for which $\tilde{S}$ multiplies distances by $|\lambda|$, and we give $\mathbf{R}^d = \bigoplus_\lambda W_\lambda$ the metric that is the supremum over the W_λ metrics. This metric is translation invariant, and hence projects under Φ to one on $\mathbf{T}^d$. Distances in $\mathbf{R}^d$ and $\mathbf{T}^d$ will be with respect to these metrics. Also, there are Haar measures ω and ω_λ on the groups $\mathbf{R}^d$ and W_λ such that $\omega = \Pi_\lambda\, \omega_\lambda$, and such that locally $\Phi\omega = \mu$.

The subspaces $W^s = \bigoplus_{|\lambda| \leq 1} W_\lambda$ and $W^u = \bigoplus_{|\lambda| > 1} W_\lambda$ are called the *weakly stable* and the *unstable subspaces* of $\tilde{S}$, respectively. The images under Φ of the cosets of W^s and the cosets of W^u form the weakly stable and unstable foliations of $\mathbf{T}^d$. A key element in our proof is the unique ergodicity of the unstable foliation in the sense of Bowen and Marcus [7]. Here this amounts to showing that if D is Jordan measurable in $\mathbf{T}^d$, then for any large enough ball B in W^u the projection under Φ of normalized Lebesgue measure on B is nearly uniformly distributed in D. The Bowen–Marcus definition of unique ergodicity of foliations can be easily checked in our case, because the leaves of the unstable foliation are the orbits of the natural action of W^u on $\mathbf{T}^d$ via translation, and the only normalized Borel measure on $\mathbf{T}^d$ invariant under this action is Haar measure, since, as we shall soon see, $\Phi(W^u)$ is dense in $\mathbf{T}^d$.

We establish unique ergodicity of the unstable foliation by using Weyl's Theorem. If σ is a probability measure on $\mathbf{R}^d$, then $\Phi\sigma$ denotes its image on $\mathbf{T}^d$. A vector $v \in \mathbf{R}^d$ is called *irrational* if its coordinates are not rationally related. A measure σ on $\mathbf{T}^d$ is *ε-uniformly distributed* in a subset D of positive measure in $\mathbf{T}^d$ if $|\sigma(D)/\mu(D) - 1| < \varepsilon$. If L is a line segment in $\mathbf{R}^d$, let σ_L denote normalized linear measure on L.

WEYL'S THEOREM. *Suppose that v is an irrational vector in $\mathbf{R}^d$, D is Jordan measurable of positive measure in $\mathbf{T}^d$, and $\varepsilon > 0$. Then there is an r_0 such that if L is a line segment in $\mathbf{R}^d$ in the direction of v of length greater than r_0, then $\Phi\sigma_L$ is ε-uniformly distributed in D.*

The proof of this is the same as that of the usual Weyl Theorem (where finite sequences of points replace line segments), namely verifying Weyl's criterion for $\Phi\sigma_L$ on the characters of $\mathbf{T}^d$ (see [15, chap. 1.2]). We point out that the result is uniform in the initial point of L, i.e. that any translate of $\Phi\sigma_L$ is also ε-uniformly distributed in D.

In order to use Weyl's Theorem, we need to know that the eigenspaces W_λ contain irrational vectors.

LEMMA 4.3. *Each W_λ contains an irrational vector v_λ.*

PROOF. If λ is real, then $v = (1, \lambda, \cdots, \lambda^{d-1})$ is in W_λ, and is irrational since the characteristic polynomial of $\tilde{S}$ is irreducible.

If λ is not real, let $w = (1, \lambda, \cdots, \lambda^{d-1})$, and put $v_\lambda = w + \bar{w} + i\xi(w - \bar{w})$, where ξ is a transcendental real number. Then $v_\lambda = (v_1, \cdots, v_d) \in W_\lambda$. If there are rationals c_j such that $c_1v_1 + \cdots + c_dv_d = 0$, then

$$\sum_{j=1}^{d} c_j(\lambda^{j-1} + \bar{\lambda}^{j-1}) + i\xi \sum_{j=1}^{d} c_j(\lambda^{j-1} - \bar{\lambda}^{j-1}) = 0.$$

Since ξ is transcendental, each sum, being algebraic, must vanish. Irreduciblity of the characteristic polynomial then forces each c_j to vanish.

If B is a measurable subset of W_λ with $0<\omega_\lambda(B)<\infty$, put $\sigma_B = \omega_\lambda(B)^{-1}(\omega_\lambda \mid B)$, where $\omega_\lambda \mid B$ denotes the restriction of ω_λ to B. The following result shows that the foliation of $\mathbf{T}^d$ into cosets of $\Phi(W_\lambda)$ is uniquely ergodic.

LEMMA 4.4. *If D is a Jordan measurable subset of $\mathbf{T}^d$ with positive measure, and $\varepsilon>0$, then there is an r_1 such that if B is a ball in W_λ of radius greater than r_1, then $\Phi\sigma_B$ is ε-uniformly distributed in D.*

PROOF. If λ is real, this follows from Weyl's Theorem and Lemma 4.3.

If λ is not real, so that dim $W_\lambda=2$, roughly we shall consider σ_B as nearly an average of line segment measures in the v_λ direction long enough for Weyl's Theorem to apply, and observe that an average of measures that are ε-uniformly distributed in D again enjoys this property.

The convolution of two integrable functions f and g on the group W_λ is defined by

$$(f*g)(t)=\int_{W_\lambda} f(u)g(t-u)\,d\omega_\lambda(u),$$

and if σ_1 and σ_2 are two finite measures on W_λ, their convolution is the measure

$$(\sigma_1*\sigma_2)(E)=\int_{W_\lambda} \sigma_2(E-u)\,d\sigma_1(u).$$

Suppose we are given D and ε as in the hypotheses. Let r_0 be supplied by Weyl's Theorem for v_λ, D, and $(\varepsilon/4)\mu(D)$. Choose r_1 large enough so that if B is a ball in W_λ of radius $r>r_1$, and B_0 is a ball of radius $r-2r_0$, then $\omega_\lambda(B_0)/\omega_\lambda(B)>(1-\varepsilon/4)\mu(D)$.

If B is now a ball of radius $r>r_1$ in W_λ, let B_0 and B_1 be balls concentric with B of radii $r-2r_0$ and $r-r_0$, respectively. Let L be the segment of length r_0 in the direction v_λ centered on the origin. If χ_B denotes the characteristic function of a subset B, and σ_B, σ_{B_0}, σ_{B_1} are abbreviated to σ, σ_0, σ_1, then since

$$\chi_{B_0}\leqq \chi_{B_1}*\chi_L\leqq \chi_B,$$

we have

$$b_0\sigma_0\leqq b_1\sigma_1*\sigma_L\leqq\sigma,$$

where $b_i=\omega_\lambda(B_i)/\omega_\lambda(B)>(1-\varepsilon/4)\mu(D)$ $(i=0,\ 1)$. Now for any $u\in W_\lambda$ we have

$$|\Phi\sigma_{L-u}(D)-\mu(D)|<(\varepsilon/4)\mu(D)$$

by our choice of r_0, and

$$|\Phi(\sigma - b_0\sigma_0)(D)| < \tfrac{1}{2}\varepsilon\mu(D)$$

since $b_0 > (1 - \varepsilon/4)\mu(D)$. Thus

$$\begin{aligned} |\Phi\sigma(D) - \mu(D)| &\leqq |\Phi\sigma(D) - b_1\Phi(\sigma_1 * \sigma_L)(D)| + b_1|\Phi(\sigma_1 * \sigma_L)(D) - \mu(D)| \\ &\quad + \tfrac{1}{4}\varepsilon\mu(D) \\ &\leqq |\Phi\sigma(D) - b_0\Phi\sigma_0(D)| + b_1\int_{B_1} |\Phi\sigma_{L-u}(D) - \mu(D)|\, d\sigma_1(u) \\ &\qquad + \tfrac{1}{4}\varepsilon\mu(D) \\ &< (\tfrac{1}{2}\varepsilon + \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon)\mu(D) \\ &= \varepsilon\mu(D). \end{aligned}$$

The next result uses this uniform distribution to prove that the toral automorphisms we are considering are Kolmogorov maps, and therefore is an alternative proof of Rohlin's theorem [31] in this case.

LEMMA 4.5. *Let S be an automorphism of $\mathbf{T}^d$ as in Theorem 4.1, α be a smooth partition and β be a Jordan measurable partition of $\mathbf{T}^d$. If $\varepsilon > 0$, there is a k_0 such that for all $k > k_0$, $m > 0$, and $t_i \in \mathbf{T}^d$ $(-m-k \leqq i \leqq -k)$, we have that ε-almost every atom of $\bigvee_{-m-k}^{-k}(S^{-i}\alpha + t_i)$ is ε-uniformly distributed in each element of β.*

PROOF. Briefly the proof runs as follows. Fixing W_λ with $|\lambda| > 1$, we show that for any choice of $u_i \in \mathbf{T}^d$, ε-almost every atom of $\bigvee_{-m}^{0}(S^{-i}\alpha + u_i)$ is approximated by an average of balls in the W_λ direction of a fixed size independent of m and the u_i. Applying S^k to these balls expands them by $|\lambda|^k$, making them large enough to apply Lemma 4.4 to obtain ε-uniform distribution. We then let $u_i = S^{-k}t_i$, so that $S^k(\bigvee_{-m}^{0} S^{-i}\alpha + u_i) = \bigvee_{-m-k}^{-k}(S^{-i}\alpha + t_i)$.

Since not all the eigenvalues λ have modulus 1 (see [14]), fix a λ with $|\lambda| > 1$. Let $B_\lambda(r)$ denote the ball in W_λ of radius r about the origin, which we identify with its image $\Phi(B_\lambda(r))$ in $\mathbf{T}^d$. The boundary of α is $\partial\alpha = \bigcup_{A\in\alpha}\partial A$, and put $\partial_\lambda(\alpha, r) = \partial\alpha + B_\lambda(r)$, the boundary of α thickened by r in the W_λ direction. Let $b = \min\{\mu(C)\colon C \in \beta\}$.

We will show that given a $\delta > 0$, there is an $r > 0$ such that for all $m > 0$ and $u_i \in \mathbf{T}^d$, the set E of t for which $t + B_\lambda(2r)$ is contained entirely in one atom of $\bigvee_{-m}^{0}(S^{-i}\alpha + u_i)$ has measure $\mu(E) > 1 - (\varepsilon b/4)^2$. Now $t + B_\lambda(2r)$ is included in one atom of $S^{-i}\alpha + u_i$ provided that $t \notin S^{-i}\partial_\lambda(\alpha, 2r|\lambda|^i) + u_i$. The complement of E is therefore

$$\bigcup_{i=-m}^{0} S^{-i}\partial_\lambda(\alpha, 2r|\lambda|^i) + u_i,$$

and has measure bounded by

$$\sum_{j=0}^{m} \mu(S^j \partial_\lambda(\alpha, 2r|\lambda|^{-j}) + u_{-j}) = \sum_{j=0}^{m} \mu(\partial_\lambda(\alpha, 2r|\lambda|^{-j}).$$

Since α is smooth, $\mu(\partial_\lambda(\alpha, s)) \leqq Ks$ for some constant K. Thus

$$\mu(\mathbf{T}^d \backslash E) \leqq \sum_{j=0}^{\infty} \mu(\partial_\lambda(\alpha, 2r|\lambda|^{-j})) \leqq 2Kr\left(\frac{1}{|\lambda|-1}\right) < \left(\frac{\varepsilon b}{4}\right)^2$$

if r is small enough.

It follows that $(\varepsilon b/4)$-almost every $A \in \bigvee_{-m}^{0}(S^{-i}\alpha + u_i)$ we have $\mu(A \cap E) > (1 - \varepsilon b/4)\mu(A)$. Let $A_0 = A \cap E$, $A_1 = A_0 + B_\lambda(r)$. Abbreviating $B_\lambda(r)$ to B, we have

$$\chi_{A_0} \leqq \chi_{A_1} * \chi_B \leqq \chi_A.$$

Thus if μ_A denotes the normalized restriction of μ to A, then

$$a_0\mu_{A_0} \leqq a_1(\mu_{A_1} * \mu_B) \leqq \mu_A,$$

where $a_i = \mu(A_i)/\mu(A) > 1 - \varepsilon b/4$. By Lemma 4.4, there is an r_1 such that the image under Φ of a ball in W_λ of radius greater than r_1 is $\varepsilon b/4$-uniformly distributed in each member of β. Choose k_0 so $r|\lambda|^{k_0} > r_1$. Then since $S^k B$ is a ball in W_λ of radius $> r_1$ for $k > k_0$, we have, using essentially the same estimates as in Lemma 4.4, that for each $C \in \beta$,

$$\begin{aligned} |\mu_{S^kA}(C) - \mu(C)| &\leqq |\mu_{S^kA}(C) - a_1\mu_{S^kA_1} * \mu_{S^kB}(C)| + a_1|\mu_{S^kA_1} * \mu_{S^kB}(C) - \mu(C)| \\ &\quad + \tfrac{1}{4}\varepsilon b \\ &\leqq |\mu_{S^kA}(C) - a_0\mu_{S^kA_0}(C)| + a_1\int |\mu_{S^kB-u}(C) - \mu(C)|\, d\mu_{S^kA_1}(u) \\ &\qquad + \tfrac{1}{4}\varepsilon b \\ &< \tfrac{1}{4}\varepsilon b + a_1\tfrac{1}{4}\varepsilon b + \tfrac{1}{4}\varepsilon b \\ &< \varepsilon\mu(C). \end{aligned}$$

We now describe the auxiliary mapping box partition mentioned above. For each λ let P_λ be a parallelogram in W_λ with one corner at the origin, so that translates of P_λ by a lattice tile W_λ. If λ is real, then P_λ is just a line segment with one endpoint being the origin. Let $C^s = \bigoplus_{|\lambda| \leqq 1} \Phi P_\lambda$, $C^u = \bigoplus_{|\lambda| \geqq 1} \Phi P_\lambda$, and $C = C^s \oplus C^u$, where the reader should recall our convention that sums indexed by λ are over only those λ with nonnegative imaginary part. We will typically be working with a C of small diameter. Then since $\bigoplus_\lambda P_\lambda$ tiles $\mathbf{R}^d$ under a lattice that is the direct sum of the tiling lattices for the P_λ, there are $b_j \in \mathbf{T}^d$ $(1 \leqq j \leqq J)$ such that the boxes $C_j = C + b_j$ $(1 \leqq j \leqq J)$ are disjoint and their complement $C_0 = \mathbf{T}^d \setminus \bigcup_1^J C_j$ has small measure. The following shows that for k large enough, most atoms in $\bigvee_{-m-k}^{-k}(S^{-i}\alpha + t_i)$ are "sheeted" in the W^u direction in each mappping box C_j for $j \geqq 1$.

LEMMA 4.6. *Let α be a smooth partition of $\mathbf{T}^d$, and β be a mapping box partition as described above. Then given $\delta > 0$, there is a k_0 such that for all $k > k_0$, all $m > 0$, and all choices of $t_i \in \mathbf{T}^d$ $(-m-k \leqq i \leqq -k)$, we have that δ-almost every $A \in \bigvee_{-m-k}^{-k}(S^{-i}\alpha + t_i)$ has a subset A_0 with $\mu(A_0) > (1-\delta)\mu(A)$, and such that for each j, $1 \leqq j \leqq J$, we have*

$$A_0 \cap C_j = A_0^s \oplus C^u,$$

where $A_0^s \in b_j + C^s$ (see Fig. 1).

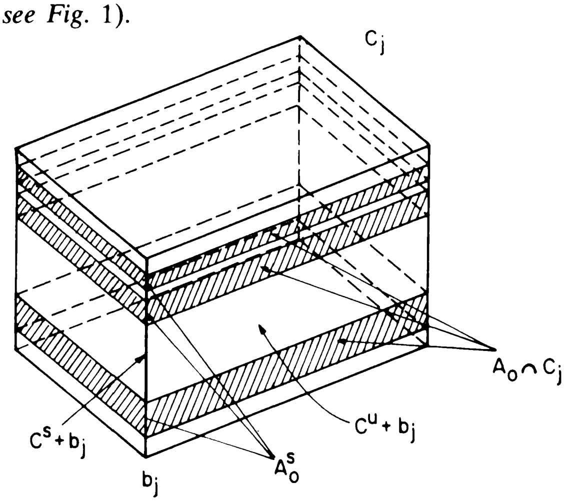

Fig. 1

PROOF. Since the proof is so similar to that of the previous lemma, we only sketch the main idea. From the proof of Lemma 4.5, we know that there is an $r > 0$ such that for almost every atom A' in $\bigvee_{-m}^{0}(S^{-i}\alpha + u_i)$ is well approximated by an average of translates of the ball B in W^u of radius r. Since S^k expands distances exponentially in W^u, except for a part E of $S^kB + t$ that lies within a fixed distance from the boundary (which therefore has relative measure in $S^kB + t$ which is exponentially small as $k \to \infty$), $S^kB + t$ is sheeted in the W^u direction in each C_j $(j \geqq 1)$. That is, for $j \geqq 1$ we have

$$[(S^kB + t)\setminus E] \cap C_j = F^s \oplus C^u,$$

where $F^s \subset b_j + C^s$. The same sheetedness persists to δ-almost all of $A = S^kA'$.

PROOF OF THEOREM 4.1. As we remarked above, by the relative very weak Bernoulli criterion and the Relative Monotone Theorem, it is enough to show that if α is a smooth partition of $\mathbf{T}^d$, then for any $t_i \in \mathbf{T}^d$ the sequence $\{S^{-i}\alpha + t_i\}$ is very weak Bernoulli.

Let $\varepsilon > 0$, and let δ be a positive number to be chosen later. There is an $\eta > 0$ such that if B is the ball $\mathbf{T}^d$ of radius η and center 0, then $\mu(\partial\alpha + B) < \delta^2$. It follows that for every $n > 0$, the set E_n of points t such that

$$\frac{1}{n}\operatorname{card}\{i: 1 \leqq i \leqq n,\ S^i t + t_{-i} \in \partial\alpha + B\} \geqq \delta \tag{4.1}$$

has $\mu(E_n) < \delta$. Choose a partition $\beta = \{C_j: 0 \leqq j \leqq J\}$ of $\mathbf{T}^d$ into mapping boxes as above such that $\operatorname{diam} C_j < \eta$ $(1 \leqq j \leqq J)$, and $\mu(C_0) < \delta$.

By Lemmas 4.5 and 4.6, there is a k such that for all $m > 0$, we have that δ-almost every $A \in \bigvee_{-m-k}^{-k}(S^{-i}\alpha + t_i)$ is δ-uniformly distributed in each C_j, and contains a subset A_0 with $\mu(A_0) > (1-\delta)\mu(A)$ and such that

$$A_0 \cap C_j = A_0^s \oplus C^u \qquad (1 \leqq j \leqq J)$$

where $A_0^s \subset b_j + C^s$. For these atoms A we define $\theta_A: A \to \mathbf{T}^d$ as follows. Let $\psi_j: A_0^s \to b_j + C^s$ be an arbitrary map. Define $\theta_A: A_0 \cap C_j \to C_j$ by $\theta_A(a_0 + b) = \psi_j(a_0) + b$ for $a_0 + b \in A_0 \cap C_j = A_0^s \oplus C^u$, so that θ_A is measure preserving on each $A_0 \cap C_j$. Define θ_A arbitrarily on $(A \setminus A_0) \cup (A \cap C_0)$. Then θ_A is not quite measure-preserving for three reasons: (1) A_0 is not quite all of A, (2) A_0 is not quite uniformly distributed in the C_j, and (3) $\bigcup_{j=1}^J C_j$ is not quite all of $\mathbf{T}^d$. However, each of these errors is bounded by δ, so that if δ is small enough, θ_A will be an ε-map for δ-almost every A.

Let B' be the ball in W^s of radius η centered on the origin. Since the eigenvalues of S are nonrepeated, $S^iB' \subset B'$ for all $i \geqq 0$. The crucial property of θ_A is that if $t \in A_0$, then $t - \theta_A t \in B'$, and hence $S^i t - S^i\theta_A t \in B'$ for $i \geqq 0$.

Choose $n > k/\varepsilon$. By (4.1) we therefore have that for $t \in A_0 \cap E_n$, the $\{S^{-i}\alpha + t_i: 1 \leqq i \leqq n\}$-names of t and $\theta_A t$ agree in more than $(1-\delta)n$ places. Since $\mu(E_n) < \delta$, the set of atoms A for which $\mu(A_0 \cap E_n) > (1-\sqrt{\delta})\mu(A)$ has measure greater than $1-\sqrt{\delta}$. Hence if δ is chosen small enough, we have by Lemma 4.2 that

$$\bar{d}[\{S^{-i}\alpha + t_i \mid A\}_1^n, \{S^{-i}\alpha + t_i\}_1^n] < \varepsilon$$

for ε-almost every $A \in \bigvee_{-m-k}^{-k}(S^{-i}\alpha + t_i)$, concluding the proof.

5. Irreducible solenoidal automorphisms

In this section we prove the Skew Product Theorem for certain automorphisms of groups whose dual is contained in a rational vector space. These automorphisms, which we have dubbed "irreducible solenoidal automorphisms", together with group shifts, form the building blocks from which any group generated by $\{T^i\gamma : i \in \mathbf{Z}\}$ is all of Γ. In the previous section we dealt mainly that preserve the Skew Product Theorem. This section logically includes the previous one. However, we hope that by emphasizing clearly there the geometry involved, the reader can follow the additional complications here more easily. The general solenoidal automorphism has been the major obstacle in extending results for automorphisms of tori to general compact groups, so that this section is the most important part of the paper.

A *solenoid* is a group G whose dual group Γ is a finite rank, torsion-free abelian group. By taking the tensor product of Γ with $\mathbf{Q}$, this definition amounts to saying that a solenoid is a group whose dual can be embedded as a subgroup of full rank in $\mathbf{Q}^d$. An automorphism S of a solenoid G has a dual automorphism T of Γ that can be uniquely extended to a rational vector space isomorphism of $\mathbf{Q}^d$. The solenoidal automorphism S is *irreducible* if the characteristic polynomial of the linear map T is irreducible, and if there is an element $\gamma \in \Gamma$ such that the group generated by $\{T^i\gamma : i \in \mathbf{Z}\}$ is all of Γ. In the previous section we dealt mainly with the irreducible solenoidal automorphism $S = C(p)'$ of $\mathbf{T}^d$, where $p(x)$ was irreducible in $\mathbf{Z}[x]$. There, heavy use was made of the fact that there was a lattice $\mathbf{Z}^d$ invariant under T. In general no such invariant lattice exists in Γ, and this complicates the geometric proof of approximate independence on the solenoid itself.

The same linear algebra used for the torus case together with the Relative Monotone and Factor Theorems show that to handle solenoidal automorphisms we need only consider the irreducible ones.

THEOREM 5.1. *The Skew Product Theorem holds for irreducible solenoidal automorphisms.*

Before beginning the proof, we first introduce some terminology and build some preliminary machinery. Some of this closely parallels the toral case, but there are novel features as well.

Let S be an irreducible solenoidal automorphism of G, T be the dual automorphism of Γ, and γ generate Γ under T. As noted before, we can consider Γ as embedded as a subgroup of $\mathbf{Q}^d$ with full rank. Since the characteristic polynomial for T is irreducible over $\mathbf{Q}$, the set $\{\gamma, T\gamma, \cdots, T^{d-1}\gamma\}$ forms a rational vector space basis for $\mathbf{Q}^d$. Let Λ be the subgroup generated by these basis elements, so that Λ forms a lattice in Γ. In the torus case $\Lambda = \Gamma$, but in general this is not true. It is helpful, though not strictly accurate, to think of Λ as being $\mathbf{Z}^d$ contained in $\Gamma = \mathbf{Q}^d$. The annihilator $H = \Lambda^\perp$ is a closed subgroup of G. It is totally disconnected because its dual Γ/Λ is a torsion group. The quotient map $\pi_H \colon G \to G/H$ is dual to the inclusion $\Lambda \subset \Gamma$. Since Λ is a d-dimensional lattice, its dual G/H is isomorphic to $\mathbf{T}^d$.

If we consider $\mathbf{Q}^d$ as embedded in $\mathbf{R}^d$, then we claim that either Γ is a lattice (corresponding to §4), or Γ is dense subgroup of $\mathbf{R}^d$ in the usual topology. For $\bar{\Gamma}$ is a closed subgroup of $\mathbf{R}^d$, and therefore has the form $W \oplus L$, where W is a real subspace and L is a lattice. Now W is T-invariant. If $W \neq 0$, it must contain a point of Γ. Irreducibility of the characteristic polynomial if T then implies that $\dim W = d$, so that $\mathbf{R}^d = W \subset \bar{\Gamma}$.

There is a natural homomorphism $\Phi \colon \mathbf{R}^d \to G$ which plays a crucial role (similar to the covering map $\Phi \colon \mathbf{R}^d \to \mathbf{T}^d$ of §4) in analysing the geometry of the action of S. Since $\Gamma \subset \mathbf{Q}^d$, a typical element of Γ has the form $\gamma = a_1\gamma + a_2 T\gamma + \cdots + a_d T^{d-1}\gamma$, where $a_i \in \mathbf{Q}$. For $t = (t_1, \cdots, t_d) \in \mathbf{R}^d$, we define $\Phi(t) \in G$ by evaluating it at γ:

$$\Phi(t)(\gamma) = \exp 2\pi i (t_1 a_1 + \cdots + t_d a_d).$$

The homomorphism Φ can be defined without recourse to coordinates by defining it to be the dual of the inclusion homomorphism $\Gamma \subset \Gamma \otimes \mathbf{R}$, where $\Gamma \otimes \mathbf{R}$ is a d-dimensional real vector space. As before, either Γ is a lattice or a dense subgroup of $\Gamma \otimes \mathbf{R}$. If Γ is a lattice, then Φ is surjective, and we are in the case of §4. Recall that Γ itself carries the discrete topology. If Γ is dense in $\Gamma \otimes \mathbf{R}$, then since a character on $\Gamma \otimes \mathbf{R}$ is determined by its values on Γ, and since any character on Γ (with the discrete topology) can be matched at any finite number of elements by a character of $\Gamma \otimes \mathbf{R}$, we see that Φ is injective with dense image. Thus in the case we are primarily concerned with , $\Phi(\mathbf{R}^d)$ is a dense, d-parameter subgroup of G. Its cosets foliate G, and the action of S on this d-dimensional foliation plays a key role.

With respect to the basis $\{\gamma, T\gamma, \cdots, T^{d-1}\gamma\}$ of $\mathbf{Q}^d$, the matrix of T is the companion matrix $C(p)$ of the characteristic polynomial $p(x)$ of T. The transpose $\tilde{S} = C(p)'$ acts on $\mathbf{R}^d$. Then

$$\Phi(\tilde{S}t) = S(\Phi t) \qquad (t \in \mathbf{R}^d),$$

which is proved by evaluating both sides on a typical element of Γ.

The homomorphism $\pi_H\Phi$: $\mathbf{R}^d \to G/H$ has kernel $\mathbf{Z}^d$, and therefore induces an isomorphism of $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$ with G/H. We use this isomorphism to identify $\mathbf{T}^d$ with G/H.

Now H is not necessarily invariant under S, for Λ is not necessarily invariant under T. However, SH projects under π_H to a finite subgroup of $\mathbf{T}^d$ with kernel $H \cap SH$. Thus if μ_H denotes Haar measure on H, then $\mu_H(H \cap SH) > 0$. A novel element in the proof is that relative to H the iterates of S on H are independent. The proof of this essentially reduces to Gauss' Lemma.

For $m \geqq 0$, let

$$H_m = H \cap SH \cap \cdots \cap S^m H \quad \text{and} \quad H_{-m} = S^{-m}H \cap S^{-m+1}H \cap \cdots \cap H.$$

LEMMA 5.2. *For nonnegative integers m and n, the subgroups H_m and H_{-n} of H are independent subsets of H with positive μ_H-measure.*

PROOF. The dual group of $H/H_{-n} \cap H_m$ is $(\Sigma_{i=-m}^{n} T^i\Lambda)/\Lambda$ (the change of sign in the indices comes from $(SH)^\perp = T^{-1}H^\perp = T^{-1}\Lambda$), which is finite since Λ has full rank in Γ. Thus $\mu_H(H_{-n} \cap H_m) > 0$, so that $\mu_H(H_{-n})$ and $\mu_H(H_m)$ are both positive.

Let $|E|$ denote the cardinality of E. Independence of H_{-n} and H_m is equivalent to $|H/H_{-n} \cap H_m| = |H/H_{-n}| \cdot |H/H_m|$, which is in turn equivalent by duality to

$$\left| \sum_{i=-m}^{n} T^i\Lambda/\Lambda \right| = \left| \sum_{i=-m}^{0} T^i\Lambda/\Lambda \right| \cdot \left| \sum_{i=0}^{n} T^i\Lambda/\Lambda \right|. \tag{5.1}$$

A polynomial that has integral coefficients whose greatest common divisor is 1 is called *primitive*. There is an integral multiple $q(x) = a_d x^d + \cdots + a_0$ of the characteristic polynomial $p(x)$ of T that is primitive. If (5.1) were false, there would be integers b_j such that

$$\sum_{j=-m}^{n+d-1} b_j T^j \gamma = 0. \tag{5.2}$$

Choose n minimal for which such a relation holds. Minimality of n guarantees that a_d does not divide b_{n+d-1}, for otherwise, using $q(T)\gamma = 0$, there would be such a relation with n replaced by $n-1$. If

$$r(x) = \sum_{j=0}^{m+n+d-1} b_{j-n} x^j,$$

then since $r(T)\gamma = 0$, $q(x)$ must divide $r(x)$ in $\mathbf{Q}[x]$, say $q(x)s(x) = r(x)$ for some $s(x) = c_{m+n-1}x^{m+n-1} + \cdots + c_0 \in \mathbf{Q}[x]$. Let c be an integer such that $cr(x)$

is primitive. Then the product $cr(x)$ of $q(x)$ and $cs(x)$ is primitive by Gauss' Lemma, which means $c = \pm 1$. Hence $b_{n+d-1} = a_d c_{m+n-1}$, contradicting our assumption that a_d does not divide b_{n+d-1}.

The subgroups H_{-n} and H_m provide a finitistic algebraic analogue on H of the stable and unstable subspaces. For if g and h are in the same coset of H_{-n}, then $S^i g$ and $S^i h$ are in the same coset of H for $0 \leqq i \leqq n$. If we let $H_{-\infty} = \bigcap_{n=1}^{\infty} H_{-n}$, $H_\infty = \bigcap_{m=1}^{\infty} H_m$, then $H = H_{-\infty} \oplus H_\infty$, and the decomposition of H into cosets of $H_{-\infty}$ and into cosets of H_∞ give the stable and unstable "foliations" of H, true analogues of the corresponding subspaces. However, we find it convenient to keep to finite intersections.

We will now use the geometry of the linear map $\tilde{S}$ via Φ to obtain the same sort of uniform distribution results as in §4. The definitions of the weakly stable and unstable subspaces, and so on, apply equally well to $\tilde{S}$ here. Thus W^s, W^u, W_λ, ω_λ, and $B_\lambda(r)$ mean the same as in §4 with respect to $\tilde{S}$. Note, however, that here W^s or W^u could be trivial. As before, there is a metric on $\mathbf{R}^d$ such that $\tilde{S}$ multiplies distances by $|\lambda|$ on W_λ, and this metric induces a compatible one on G via Φ. The proof of Lemma 4.3 applies since $p(x)$ is irreducible to show that each W_λ contains an irrational vector v_λ.

The next two lemmas are preparation for the case $W^u \neq 0$. The case when all eigenvalues of $\tilde{S}$ have modulus 1 is handled separately.

Let α' be a smooth partition of $\mathbf{T}^d$ into connected sets, and $\alpha = \pi_H^{-1}(\alpha')$. We call such an α a *smooth partition of* G. The first result shows that most atoms of $\bigvee_{-m}^{0}(S^{-i}\alpha + g_i)$ are "thick" in the unstable direction. Recall that $B_\lambda(r)$ is the ball in W_λ of radius r about 0, and we let $B^u(r) = \bigoplus_{|\lambda|>1} B_\lambda(r)$. We identify $B_\lambda(r)$ and $B^u(r)$ with their images in G under Φ.

A subset E of G is H_m*-saturated* if $E + H_m = H_m$, that is, E is a union of cosets of H_m.

Lemma 5.3. *Let α be a smooth partition of G, and suppose $W^u \neq 0$. Given $\eta > 0$, there is an $r > 0$ such that for all $m > 0$ and all g_i $(-m \leqq i \leqq 0)$, η-almost every atom $A \in \bigvee_{-m}^{0}(S^{-i}\alpha + g_i)$ contains an H_m-saturated subset A_0 with $\mu(A_0) > (1-\eta)\mu(A)$ and $A_0 + B^u(2r) \subset A$.*

Proof. The proof consists of the same sort of estimate as in the proof of Lemma 4.5 of the measure of the set of points exponentially close to the boundary of α'.

An atom $A_k \in \alpha$ has the form $\pi_H^{-1}(A'_k)$, where $A'_k \in \alpha'$ is connected with piecewise smooth boundary. Ignoring the null set $\partial\alpha'$, there is an open connected set $D_k \subset \mathbf{R}^d$ with piecewise smooth boundary, unique up to translation by connectedness of A'_k, such that $\pi_H\Phi(D_k) = A'_k$. Hence $A_k = \Phi(D_k)\oplus H$. Since $\Phi(\mathbf{R}^d)$ is dense in G, for arbitrary $g_i \in G$ there is a $t_i \in \mathbf{R}^d$ such that $A_k + S^{-i}g_i = \Phi(D_k + t_i)\oplus H$ for every k. Thus an atom $A \in \bigvee_{-m}^{0}(S^{-i}\alpha + g_i)$ has the form

$$A = \bigcap_{i=0}^{m} S^i(\Phi(D_{k_i} + t_{-i})\oplus H) = \bigcap_{i=0}^{m} (\Phi(\tilde{S}^i D_{k_i} + \tilde{S}^i t_{-i})\oplus S^i H).$$

Now S^iH is a finite disjoint union of cosets of H_m. Hence each term in the intersection is a finite disjoint union of H_m-saturated sets of the form

$$\Phi(\tilde{S}^i D_{k_i} + u_i)\oplus H_m.$$

Therefore the intersection A itself is a finite disjoint union of sets of the form

$$\bigcap_{i=0}^{m} \Phi(\tilde{S}^i D_{k_i} + u_i)\oplus H_m,$$

for various choices of the u_i.

Let $\rho = \min\{|\lambda_i| : |\lambda_i| > 1\} > 1$. Since each D_k has smooth boundary, and since $\tilde{S}$ expands distances in the W^u direction by a factor of at least ρ, the proportion of $S^iD_k + u_i$ lying a distance less than $2r$ from its boundary in the W^u direction is bounded by a constant times $r\rho^{-i}$. It follows that the measure of the set of points $g \in G$ such that $g + B^u(2r)\oplus H_m$ is not completely contained in a single atom of $S^{-i}\alpha + g_i$ is bounded by $Kr\rho^{-i}$ for some constant K. Hence the set E of g for which $g + B^u(2r)\oplus H_m$ is in one atom of $\bigvee_{-m}^{0}(S^{-i}\alpha + g_i)$ has measure greater than

$$1 - \sum_{i=0}^{m} Kr\rho^{-i} > 1 - \sum_{i=0}^{\infty} Kr\rho^{-i} = 1 - \frac{Kr}{1-\rho} > 1 - \eta^2,$$

provided r is small enough.

Thus for η-almost every $A \in \bigvee_{-m}^{0}(S^{-i}\alpha + g_i)$, we have that if $A_0 = A \cap E$, then $\mu(A_0) > (1-\eta)\mu(A)$. Since A and E are H_m-saturated, so is their intersection A_0. Finally, it follows from the definition of E that $A_0 + B^u(2r) \subset A$.

The subspaces W_λ of $\mathbf{R}^d$ project under $\pi_H\Phi$ to $\mathbf{T}^d$, and with respect to their images construct a mapping box partition β' of $\mathbf{T}^d$ exactly as in §4. We remind the reader that $\beta' = \{C'_0, C'_1, \cdots, C'_J\}$, where $C'_j = C' + t_j$ $(1 \leqq j \leqq J)$, $C' \doteq \pi_H\Phi(C^s \oplus C^u)$, where C^s and C^u are parallelograms in W^s and W^u with small diameter. Let $\beta = \pi_H^{-1}(\beta')$. We shall identify C^s and C^u with their images in G under Φ. Thus if b_j is chosen such that $\pi_H(b_j) = t_j$, then $\pi_H^{-1}(C'_j) = C_j = C^s \oplus C^u \oplus H + b_j$.

The next result, containing the analogues of Lemmas 4.5 and 4.6, shows that with respect to β most atoms of $\bigvee_{-m-k}^{-k}(S^{-i}\alpha + g_i)$ are nearly uniformly distributed and "sheeted" in the W^u direction.

LEMMA 5.4. *Let α be a smooth partition of G, β be a mapping box partition as described above, and suppose that $W^u \neq 0$. Given $\delta > 0$, there is a $k > 0$ such that for all $m > 0$ and all g_i $(-m-k \leqq i \leqq -k)$, we have that δ-almost every $A \in \bigvee_{-m-k}^{-k}(S^{-i}\alpha + g_i)$ contains an H_{m+k}-saturated set A' such that $\mu(A') > (1-\delta)\mu(A)$, A' is δ-uniformly distributed in every $C_j \in \beta$, and $A' \cap C_j = A^j \oplus C^u$, where $A^j \subset C^s \oplus H + b_j$ (see Fig. 2).*

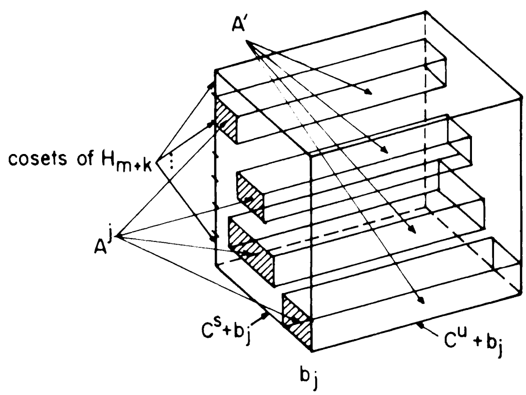

Fig. 2

PROOF. Let η be a small positive number to be chosen later. By the previous lemma, there is an $r > 0$ such that for all $m > 0$ and all h_i $(-m \leqq i \leqq 0)$, η-almost every $A \in \bigvee_{-m}^{0}(S^{-i}\alpha + h_i)$ contains an H_m-saturated subset A_0 with $\mu(A_0) > (1-\eta)\mu(A)$ and $A_0 + B^u(2r) \subset A$. Fix a λ with $|\lambda| > 1$. If $A_1 = A_0 + B_\lambda(r)$, then $A_0 \subset A_1 \subset A$, and

$$\chi_{A_0} \leqq \chi_{A_1} * \chi_{B_\lambda(r)} \leqq \chi_A .$$

Applying S^k to this gives

$$\chi_{S^kA_0} \leqq \chi_{S^kA_1} * \chi_{B_\lambda(r|\lambda|^k)} \leqq \chi_{S^kA} . \tag{5.3}$$

By Lemma 4.4, there is an r_1 such that if B is a ball in W_λ of radius greater than r_1, then $\Phi\sigma_B$ is η-uniformly distributed in β. Thus for all k large enough so that $r|\lambda|^k > r_1$, the central term in (5.3) is an average of measures that are η-uniformly distributed in β. Since $\mu(A_0) > (1-\eta)\mu(A)$, the same proof as in Lemma 4.5 shows that S^kA_0 and S^kA are 4η-uniformly distributed in β.

Let $\rho = \min\{|\lambda| : |\lambda| > 1\}$. Suppose that $g \in A_0$, and $S^kg \in C_j$ for some $j \geqq 1$.

Then $S^k g = b^s + b^u + b_j$, where $b^s \in C^s$ and $b^u \in C^u$. Since $g + B^u(r) \subset A$, it follows that $S^k g + B^u(r\rho^k) \subset S^k A$. Hence if k is large enough so that $r\rho^k > 2\max\{\operatorname{diam} C'_j : 1 \leqq j \leqq J\}$, we have that $S^k g \in b^s + C^u + b_j \subset A$, where the central term is the W^u sheet through $S^k g$ in C_j. Thus for all large enough k there is a subset A' of A containing A_0 such that $S^k A'$ has the required form. Since both $S^k A_0$ and $S^k A$ are 4η-uniformly distributed in β, A' will be δ-uniformly distributed in β if η is small enough.

The preparations are complete.

Proof of Theorem 5.1. Since Λ generates the dual group Γ under T, the σ-subalgebra $\mathcal{M}(H)$ generates $\mathcal{M}$ under S. Taking an increasmg sequence of smooth partitions of G whose span is $\mathcal{M}(H)$, the Relative Monotone Theorem shows that it is only necessary to show that $\{S^{-i}\alpha + g_i\}$ is very weak Bernoulli for a smooth partition α. The proof of this breaks naturally into three cases.

Expansive case. This occurs when all the eigenvalues have modulus greater than 1. Suppose that α is a smooth partition of G and fix $g_i \in G$. Let $\varepsilon > 0$, and let δ be a small positive number to be determined later. For the expansive case $W^s = 0$, and for the mapping box partition β' we take the C'_j $(j \geqq 1)$ to be parallelograms in $\mathbf{T}^d$ with small diameter, and such that $\mu(C_0) < \delta$.

By Lemma 5.4 there is a k such that for all $m > 0$, δ-almost every $A \in \bigvee_{-m-k}^{-k}(S^{-i}\alpha + g_i)$ contains an H_{m+k}-saturated subset A' such that $\mu(A') > (1-\delta)\mu(A)$, A' is δ-uniformly distributed in β, and for $1 \leqq j \leqq J$ we have $A' \cap C_j = A^j \oplus C^u$, where $A^j \subset H + b_j$. Since A' is H_{m+k}-saturated, A^j is a union of cosets of H_{m+k}.

Let $n > k/\varepsilon$. Then since H_{-n} and H_{m+k} are independent subsets of H, so that their coset partitions of H are independent, there is a measure-preserving map $\psi_j : A^j \to H + b_j$ such that $g - \psi_j(g) \in H_{-n}$. This property of ψ_j means that $S^i g$ and $S^i \psi_j(g)$ are in the same coset of H, and hence have the same α-name, for $0 \leqq i \leqq n$ (i.e. a point and its image under ψ_j are on the same "contracting fiber" in H). Define θ_A on $A' \cap C_j = A^j \oplus C^u$ by

$$\theta_A(g + b^u) = \psi_j(g) + b^u \qquad (g \in A^j, \quad b^u \in C^u),$$

and arbitrarily on $(A \backslash A') \cup C_0$. Now θ_A is measure-preserving on each $A' \cap C_j$ $(j \geqq 1)$. Since $\mu(A') > (1-\delta)\mu(A)$, $\mu(C_0) < \delta$, and A' is δ-uniformly distributed in β, θ_A will be an $(\varepsilon/16)$-map for δ-almost every A if δ is small enough. Off $(A \backslash A') \cup C_0$, a set of measure less than $[\delta + \delta(1+\delta)]\mu(A)$, the $\{S^{-i}\alpha + g_i\}_1^n$-names of g and $\theta_A(g)$ agree exactly. By Lemma 4.2, it follows that if δ is small enough, then for ε-almost every $A \in \bigvee_{-m-k}^{-k}(S^{-i}\alpha + g_i)$ we have

$$\bar{d}[\{S^{-i}\alpha + g_i \mid A\}_1^n, \{S^{-i}\alpha + g_i\}] < \varepsilon. \tag{5.4}$$

Mixed case. This is the case when some eigenvalues have modulus $\leqq 1$, and some have modulus > 1. Note that if all eigenvalues have modulus $\leqq 1$, and some have modulus < 1, then replacing S by S^{-1} puts into either the expansive or mixed case.

Let $\alpha = \pi_H^{-1}(\alpha')$ be a smooth partition of G, let $\varepsilon > 0$, and let δ be a small positive number to be determined later. Choose $\eta > 0$ such that $\mu(\partial\alpha' + B(\eta)) < \delta^4$, where $B(\eta)$ denotes the ball of radius η in $\mathbf{T}^d$ about 0. Let $\beta = \{C_j : 0 \leqq j \leqq J\}$ be a mapping box partition of G as described above such that diam $\pi_H C_j < \eta$ $(1 \leqq j \leqq J)$ and $\mu(C_0) < \delta$. By Lemma 5.4, there is a $k > 0$ such that for all $m > 0$, δ-almost every $A \in \bigvee_{-m-k}^{-k}(S^{-i}\alpha + g_i)$ has an H_{m+k}-saturated subset A' such that $\mu(A') > (1-\delta)\mu(A)$, A' is δ-uniformly distributed in β, and $A' \cap C_j = A^j \oplus C^u$ $(1 \leqq j \leqq J)$, where $A^j \subset C^s \oplus H + b_j$.

Denote by $B^s(\delta)$ the ball in W^s of radius δ about 0, identified as usual with its image $\Phi B^s(\delta)$ in G. For $1 \leqq j \leqq J$ we will construct a measure-preserving map $\psi_j : A^j \to \bar{A}^j \subset C^s \oplus H + b_j$ such that $g - \psi_j(g) \in B^s(\delta)$, and if $b \in C^s + b_j$, then $H + b$ intersects $\bar{A}^j$ in exactly one coset of H_{m+k}. Figure 3 should make clear the idea behind our admittedly clumsy construction. We can write H as a finite disjoint union $\bigcup_q (H_{m+k} + h_q)$, where h_q are coset representatives of the elements of H/H_{m+k}. Then $C^s \oplus H = \bigcup_q (C^s \oplus H_{m+k} + h_q)$. Since A^j is H_{m+k}-saturated, $A^j \cap (C^s \oplus H_{m+k} + b_j + h_q) = E_q \oplus H_{m+k}$, where E_q is a measurable

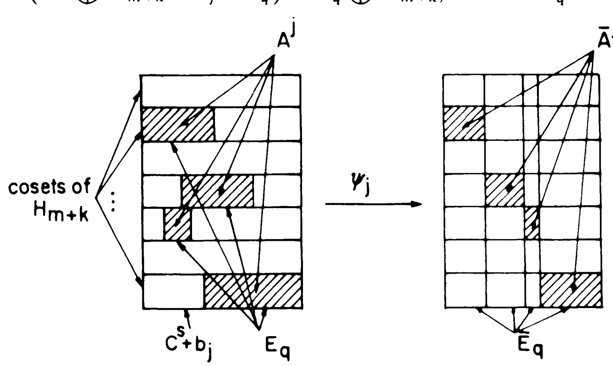

Fig. 3

subset of $C^s + b_j + h_q$. Let ω^s be Lebesgue measure on C^s or any translate. Partition $C^s + b_j$ into sets $\bar{E}_q$ such that

$$\frac{\omega^s(\bar{E}_q)}{\omega^s(C^s + b_j)} = \frac{\omega^s(E_q)}{\sum_r \omega^s(E_r)} .$$

Then there is a measure-preserving map ψ_j': $\bigcup_q E_q \to \bigcup_q(\bar{E}_q + h_q)$ such that $\psi_j'(E_q) = \bar{E}_q + h_q$. It follows from $\operatorname{diam} C_j < \delta$ that $g - \psi_j'(g) \in B^s(\delta)$. For a typical element $e + h$ of $E_q \oplus H_{m+k}$ define $\psi_j(e + h) = \psi_j'(e) + h$. This defines ψ_j on A^j.

We can now produce $\theta_A: A \to G$ for δ-almost every A. Fix $n > k/\varepsilon$. Using the ψ_j we first make a preliminary rearrangement θ_A' of A' in each C_j in order to have each coset of H intersect $\theta_A'(A')$ in one coset of H_{m+k}. We then expand each coset of H_{m+k} along the "contracting fibers", namely cosets of H_{-n}, by a map θ_A''. Then θ_A will be the composition of θ_A' and θ_A''.

On a typical element $a + b \in A^j \oplus C^u = A' \cap C_j$, define $\theta_A'(a + b) = \psi_j(a) + b$, so that

$$\theta_A'(A^j \oplus C^u) = \bar{A}^j + C^u = \bigcup_q \bar{E}_q \oplus H_{m+k} \oplus C^u,$$

where $\bar{E}_q \subset C^s + h_q + b_j$ as above. Define θ_A' arbitrarily on $(A \setminus A') \cup (A \cap C_0)$. Then θ_A' is measure-preserving on $A' \cap C_j$ $(j \geqq 1)$, and each coset of H not in C_0 intersects $\theta_A'(A')$ in exactly one coset of H_{m+k}. Since H_{m+k} is independent of H_{-n}, there is a measure-preserving map ξ: $H_{m+k} \to H$ such that $h - \xi(h) \in H_{-n}$. Define θ_A'' on a typical element $e + h + b$ of $\bar{E}_q \oplus H_{m+k} \oplus C^u$ by $\theta_A''(e + h + b) = e + \xi(h) + b$, and arbitrarily on $\theta_A'[(A \setminus A') \cup (A \cap C_0)]$. Since $\theta_A'(A' \setminus C_0)$ intersects each coset of H in C_j $(j \geqq 1)$ in exactly one coset of H_{m+k}, it follows that I loathe writing up this fulsome material and that θ_A'' is measure-preserving on each $\bar{A}^j \oplus C^u = \theta_A'(A^j \oplus C^u)$. Let $\theta_A = \theta_A''\theta_A'$. Then for $j \geqq 1$ we have that the restriction of θ_A to $A' \cap C_j$ is a measure-preserving map to C_j.

Since $\mu(A') > (1 - \delta)\mu(A)$, A' is δ-uniformly distributed in β, and θ_A is measure-preserving on $A' \cap C_j$ $(j \geqq 1)$, it follows that θ_A is an ε-map if δ is chosen small enough.

Since $\mu(\partial\alpha' + B(\eta)) < \delta^4$, the set E of points $g \in G$ such that

$$\frac{1}{n} \mid \{i: 1 \leqq i \leqq n,\ S^i g + g_{-i} \in \pi_H^{-1}(\partial\alpha' + B(\eta))\} \mid > \delta^2$$

has $\mu(E) < \delta^2$. Hence for δ-almost every A, $\mu(A \cap E) > (1 - \delta)\mu(A)$. Thus for 2δ-almost every A, $\mu(A' \cap E) > (1 - 2\delta)\mu(A)$. If $g \in (A' \setminus C_0) \cap E$, then since $g - \theta_A'(g) \in B^s(\eta)$ and S does not expand distances in $B^s(\eta)$, g and $\theta_A'(g)$ have the same $\{S^{-i}\alpha + g_i\}_1^n$-names. Since $\theta_A''(\theta_A' g) - \theta_A' g \in H_{-n}$, we have $S^i\theta_A''(\theta_A' g) - S^i\theta_A' g \in H$ for $0 \leqq i \leqq n$, and hence $\theta_A''(\theta_A' g)$ and $\theta_A' g$ have the same $\{S^{-i}\alpha + g_i\}_1^n$-names. Thus for $g \in (A' \setminus C_0) \cap E$, the $\{S^{-i}\alpha + g_i\}_1^n$-names of g and $\theta_A g$ agree. By Lemma 4.2, if δ is small enough we obtain that for ε-almost every

$A \in \bigvee_{-m-k}^{-k}(S^{-i}\alpha + g_i)$, the $\bar{d}$ inequality (5.4) holds, concluding the proof of this case.

Central Case. Here we treat the case when all eigenvalues of T have modulus 1. This does not occur for toral automorphisms (Kronecker [14] proved over a century ago that an algebraic integer all of whose conjugates are on the unit circle must be a root of unity), but can on solenoids. For example, the companion matrix of $x^2 + \frac{1}{2}x + 1$ has eigenvalues $(-1 \pm i\sqrt{15})/4$, each of modulus 1.

The geometry of stable and unstable subspaces is now unavailable, but is replaced by a multiplicity of images which was a mere nuisance in the previous two cases. The uniform distribution of these images is the content of the following result.

LEMMA 5.5. *Let β be a smooth partition of $\mathbf{T}^d$. Given $\delta > 0$, there are arbitrarily large $k > 0$ such that for all $m \geqq 0$, every translate of the finite subgroup $\pi_H(S^k H_m)$ is δ-uniformly distributed in β.*

PROOF. Uniform distribution of a finite set refers, of course, to normalized counting measure on the set. We first show that $\pi_H(S^k H_m)$ is the same for all $m \geqq 0$. This is a consequence of the independence properties of the $S^i H$. Since $S^k H_m \subset S^k H$, it suffices to show that $\pi_H(S^k H_m) \supset \pi_H(S^k H)$. Since $S^{-k}H$ is a union of cosets of H_{-k}, Lemma 4.2 implies that $S^{-k}H$ is independent of H_m. Hence if $g \in H$, then $H_m \cap (S^{-k}H + g) \neq \varnothing$ since it has positive μ_H-measure. If g' is an element of this intersection, then $S^k g' + H = S^k g + H$, so that $\pi_H(S^k H_m) \supset \pi_H(S^k H)$.

Let β and $\delta > 0$ be given. Choose $\eta > 0$ so that $\mu(\partial\beta + B(\eta)) < \frac{1}{2}\delta\mu(C_j)$ for each $C_j \in \beta$. We next show that there are arbitrarily large k for which $\pi_H(S^k H)$ contains d vectors spanning a parallelogram with positive measure and with diameter less than η. For if the eigenvalues λ_i of T have modulus 1, by Dirichlet's Theorem there are arbitrarily large k for which $|\lambda_i^k - 1| < \eta$ holds simultaneously for $1 \leqq i \leqq d$. Then if $\{e_1, \cdots, e_d\}$ is the standard basis for $\mathbf{R}^d$, since $\tilde{S}^k - I$ is nonsingular (none of the λ_i are roots of unity), the vectors $\{\tilde{S}^k e_i - e_i : 1 \leqq i \leqq d\}$ span a parallelogram $\tilde{P}$ of positive Lebesgue measure such that $\operatorname{diam} \tilde{P} < \eta$. Since $\pi_H \Phi(\tilde{S}^k e_i - e_i) = \pi_H(S^k \Phi(e_i) - \Phi(e_i)) \in \pi_H(S^k H)$ and $\pi_H \Phi$ is a local isomorphism, $P = \pi_H \Phi(\tilde{P})$ satisfies our requirements.

There is a finite subset K of $\pi_H(S^k H)$ such that $\{P + k : k \in K\}$ is disjoint and the complement E of $\bigcup_{k \in K}(P + k)$ has $\mu(E) < 2^d \eta$; this is because $\tilde{P}$ tiles the unit cube in $\mathbf{R}^d$ to within η of its 2^d faces. For each $t \in \mathbf{T}^d$, off the set $(\partial\beta + B(\eta)) \cup (E + t)$ each C_j is a union of parallelograms $P + k + t$ for k in some subset of K, and so if $\eta < 2^{-d-1}\min_j \mu(C_j)$, each C_j is approximated to within $\delta\mu(C_j)$ by a union of the $P+k+t$. Now each $P+k+t$ contains precisely the same number of elements of $\pi_H(S^kH)$. Hence $\pi_H(S^kH)$ itself is δ-uniformly distributed in β.

We are now ready to prove Theorem 5.1 for the central case. Let α, β', β, δ, and η be the same as at the beginning of the proof of the mixed case. Choose k by Lemma 5.5 such that for all $m > 0$ every translate of $\pi_H(S^kH_m)$ is δ-uniformly distributed in β'. Now if $A \in \mathrm{V}^{0}_{-m}(S^{-i}\alpha + g_i)$, then $A + H_m = A$ since A is H_m-saturated. Thus $\chi_A * \chi_{H_m} = \chi_A$. Applying S^k gives

$$\chi_{S^kA} * \chi_{S^kH_m} = \chi_{S^kA}.$$

Since every translate of S^kH_m is δ-uniformly distributed in β, the same property holds for an average S^kA of them. The proof is now completed exactly as in the mixed case, where now $W^u = 0$ means that sheetedness with respect to C^u is trivially satisfied.

6. Totally disconnected groups

We shall prove here the Skew Product Theorem for automorphisms of compact abelian totally disconnected groups. The proof uses algebraic ideas from [18]. Recall from §2 that if S is an automorphism of G, H is an invariant subgroup of G under S, and $V = U\times_\phi S$, then V/H denotes the skew product $U\times_{\bar\phi} S_{G/H}$, where $\bar\phi = \pi_H\phi$, and that V is a skew product of V/H with S_H. We identify V/H with V on the factor $\mathcal{N}\otimes\mathcal{M}(H)$.

THEOREM 6.1. *The Skew Product Theorem holds for ergodic automorphisms of compact abelian totally disconnected groups.*

PROOF. A compact abelain group is totally disconnected if and only if its dual is a torsion group. We shall first prove the result when the dual is annihilated by multiplication by a prime p (i.e. the dual is a *p-group*), and then obtain the general case from this.

So assume that S is an ergodic automorphism of a group G whose dual Ω is a p-group. If $\mathbf{Z}_p$ denotes the field $\mathbf{Z}/p\mathbf{Z}$, and R is the ring $\mathbf{Z}_p[x, x^{-1}]$ of polynomials in x and x^{-1} with coefficients in $\mathbf{Z}_p$, then R acts on Ω via the dual automorphism T of S by

$$\left(\sum_{j=-m}^{n} a_j x^j\right)\cdot\omega = \sum_{j=-m}^{n} a_j T^j\omega \qquad (\omega\in\Omega).$$

Since $\mathbf{Z}_p$ is a field, R is a principal ideal domain (this is where primality of p

enters). Let $\{\Omega_j\}_1^\infty$ be an increasing sequence of finitely generated R-submodules of Ω whose union is Ω. By the fundamental theorem for finitely generated modules over a principal ideal domain (see, for example, [19]), each $\Omega_j = E_j \oplus F_j$, where E_j is torsion and F_j is free over R. Now R-submodules of finitely generated R-modules are again finitely generated, and a finitely generated torsion R-submodule is finite. Since T is aperiodic on Ω, this forces each $E_j = 0$. Thus $\Omega_j = F_j = R\omega_1 \oplus \cdots \oplus R\omega_{r_j}$, where $\{\omega_1, \cdots, \omega_{r_j}\}$ is a free R-basis for F_j. Hence the factor automorphism S on $\mathcal{M}(\Omega_j^\perp)$ is the group shift on $\mathbf{Z}_p^{r_j}$. If follows from Theorem 3.1 that if $U \times_\phi S$ is a skew product acting on $\mathcal{N} \otimes \mathcal{M}$, then $\mathcal{N} \times \mathcal{M}(\Omega_j^\perp)$ is Bernoulli mod $\mathcal{N}$. Since $\Omega_j \nearrow \Omega$, we have $\mathcal{M}(\Omega_j^\perp) \nearrow \mathcal{M}$, and an application of the Relative Monotone Theorem shows that $\mathcal{N} \otimes \mathcal{M}$ is Bernoulli mod $\mathcal{N}$.

We now turn to the general case. Let S be an ergodic automorphism of G with dual automorphism T of the torsion dual Γ. The subgroup $\Gamma(n) = \{\gamma \in \Gamma : n\gamma = 0\}$ is T-invariant. Let $V = U \times_\phi S$ be a skew product acting on $\mathcal{N} \otimes \mathcal{M}$. Since $\Gamma(m!) \nearrow \Gamma$, which implies $\mathcal{N} \otimes \mathcal{M}(\Gamma(m!)^\perp) \nearrow \mathcal{N} \otimes \mathcal{M}$, it suffices by the Relative Monotone Theorem to show that $\mathcal{N} \otimes \mathcal{M}(\Gamma(m)^\perp)$ is Bernoulli mod $\mathcal{N}$ for every m. If this were not so, take m to be minimal for which this fails. Choose a prime p dividing m. By minimality of m, $\mathcal{N} \otimes \mathcal{M}(\Gamma(m/p)^\perp)$ is Bernoulli mod $\mathcal{N}$. Now $V/\Gamma(m)^\perp$ is a skew product of $V/\Gamma(m/p)^\perp$ with the restriction of S to $\Gamma(m/p)^\perp$ (the annihilator taken with respect to $\Gamma(m)^\perp$). The dual of $\Gamma(m/p)^\perp$ is $\Omega = \Gamma(m)/\Gamma(m/p)$, which is a p-group. Also, T is aperiodic on Ω, for if $\gamma \in \Gamma(m)$ has $T^k\gamma = \gamma + \gamma'$, $\gamma' \in \Gamma(m/p)$, then $T^k(m/p)\gamma = (m/p)\gamma$. Aperiodicity of T implies $(m/p)\gamma = 0$, whence $\gamma \in \Gamma(m/p)$. Thus $V/\Gamma(m)^\perp$ is a skew product of a Bernoulli factor mod $\mathcal{N}$ with an ergodic automorphism of a group whose dual is a p-group. By the first part of the proof, $\mathcal{N} \otimes \mathcal{M}(\Gamma(m)^\perp)$ is Bernoulli mod $\mathcal{N} \otimes \mathcal{M}(\Gamma(m/p)^\perp)$, which in turn is by our assumption Bernoulli mod $\mathcal{N}$. Thus $\mathcal{N} \otimes \mathcal{M}(\Gamma(m)^\perp)$ is Bernoulli mod $\mathcal{N}$, and this contradiction completes the proof.

The reader may be tempted to think that with more algebraic care, the use of results from ergodic theory such as the Relative Monotone Theorem could be circumvented. For example, in the first part of the proof if it were true that the torsion-free R-module Ω were actually free (true if Ω is finitely generated), then the proof would be finished.

However, it is not always possible to obtain algebraically an independent generator, as the following example shows. Let Γ be the vector space over $\mathbf{Z}_2$ with basis $\{\gamma_i : i \in \mathbf{Z}\} \cup \{\xi_j : j \geqq 1\}$. Define an automorphism T of Γ by $T\gamma_i = \gamma_{i+1}$, $T\xi_1 = \xi_1 + \gamma_0$, and $T\xi_j = \xi_j + \xi_{j-1}$ $(j \geqq 2)$. The subgroup Γ_j generated by ξ_j under T contains ξ_i for $i \leqq j$ and all the γ_i. Each Γ_j is free over $R = \mathbf{Z}_2[x, x^{-1}]$ with basis

$\{\xi_i\}$, and any finitely generated R-submodule is contained in some Γ_j. The dual automorphism S on the factor $\mathcal{M}(\Gamma_j^\perp)$ has independent two-set generator P_j induced by the coordinate ξ_j. But P_i is independent of P_j for $i \neq j$ since they depend on different coordinates. This means that although for each j, $\mathcal{M}(\Gamma_j^\perp)$ has an independent generator P_j, these P_j do not converge and so do not allow us to conclude anything about the limiting σ-algebra $\mathcal{M}$. The proof of the absolute Monotone Theorem shows that by using ergodic theory, one can arrange to have independent generators P'_j of $\mathcal{M}(\Gamma_j^\perp)$ which converge arbitrarily rapidly to a partition P, which will therefore be an independent generator for $\mathcal{M}$. This cannot be done algebraically.

7. General groups

The results of the previous two sections are assembled here into a proof of our results for general compact abelian groups. We also indicate how they can be extended to nonabelian compact groups.

As noted in §2, both the Automorphism and Splitting Theorems follow from the following statement, repeated here for reference.

Skew Product Theorem. *Skew products with ergodic automorphisms of compact abelian groups are Bernoulli* mod *the base factor.*

Proof. The basic strategy is to build up a general ergodic automorphism from those of solenoids and totally disconnected groups by using duality and the Relative Factor and Relative Monotone Theorems.

Let S be an ergodic automorphism of the compact abelian group G, with dual automorphism T of the discrete dual Γ. As usual, $\mathcal{M}$ denotes the Borel sets on G. Let U be a map of $(X, \mathcal{N}, \nu)$, and let $V = U \times_\phi S$.

Denote by Ω the torsion subgroup of Γ, that is, the subgroup of elements with finite order. Clearly Ω is T-invariant. Let H be the annihilator of Ω. Then V/H is a skew product of U with the ergodic automorphism $S_{G/H}$ of G/H, whose dual Ω is torsion. Thus by the previous section, $\mathcal{N} \otimes \mathcal{M}(H)$ is Bernoulli mod $\mathcal{N}$. Now V is skew product of V/H with S_H. The dual of H is the torsion-free group Γ/Ω. Also, T is aperiodic on Γ/Ω, for if $T^k\gamma = \gamma + \omega$ for $\omega \in \Omega$, then $n\omega = 0$ for some n. Hence $T^k(n\gamma) = n\gamma$, so aperiodicity of T shows $n\gamma = 0$, that is $\gamma \in \Omega$. Hence S_H is ergodic, and we are reduced to proving the Skew Product Theorem for groups whose dual is torsion-free. For then $\mathcal{N} \otimes \mathcal{M}$ would be Bernoulli mod $\mathcal{N} \otimes \mathcal{M}(H)$, which we already know to be Bernoulli mod $\mathcal{N}$. By our remark in §2, it would follow that $\mathcal{N} \otimes \mathcal{M}$ is Bernoulli mod $\mathcal{N}$.

Thus we assume that Γ is torsion-free. The tensor product $\Gamma \otimes \mathbf{Q}$ consists of sums of the form $\gamma_1 \otimes q_1 + \cdots + \gamma_k \otimes q_k$, where $\gamma_i \in \Gamma$, $q_i \in \mathbf{Q}$. The map $\gamma \to \gamma \otimes 1$ is an embedding of Γ into $\Gamma \otimes \mathbf{Q}$ since Γ is torsion-free (see [10]). $\Gamma \otimes \mathbf{Q}$ is a rational vector space, and T extends to a rational linear transformation of $\Gamma \otimes \mathbf{Q}$ by $T(\gamma \otimes q) = T\gamma \otimes q$. This extension is an automorphism of $\Gamma \otimes \mathbf{Q}$. For T is surjective since $\Sigma \gamma_i \otimes q_i = T(\Sigma T^{-1}\gamma_i \otimes q_i)$, and T is injective because if $T(\Sigma \gamma_i \otimes q_i) = 0$, then for any integer m such that $mq_i \in \mathbf{Z}$ we have

$$0 = T(\Sigma \gamma_i \otimes mq_i) = T(\Sigma (mq_i)\gamma_i) \otimes 1,$$

which implies $\Sigma (mq_i)\gamma_i = 0$, so that $m \Sigma \gamma_i \otimes q_i = 0$, or $\Sigma \gamma_i \otimes q_i = 0$.

Let $\bar{G}$ be the dual of $\Gamma \otimes \mathbf{Q}$, $\bar{S}$ the dual of the extension of T to $\Gamma \otimes \mathbf{Q}$, $\pi_G: \bar{G} \to G$ the quotient dual to $\Gamma \subset \Gamma \otimes \mathbf{Q}$, and $\bar{\phi}: X \to \bar{G}$ any measurable function such that $\pi_G \bar{\phi} = \phi$. Then V is the factor $U \times_{\bar{\phi}} \bar{S}/\ker \pi_G$, so by the Relative Factor Theorem it is enough to restrict our attention to the case when Γ is a rational vector space.

With this assumption on Γ, we can make it into a module over $R = \mathbf{Q}[x, x^{-1}]$ under the action of T just as in §6, where we have replaced the coefficient field $\mathbf{Z}_p$ with $\mathbf{Q}$. The ring R is again a principal ideal domain; indeed our embedding of Γ into a rational vector space was necessitated by $\mathbf{Z}[x, x^{-1}]$ not being principal.

Let $\{\Gamma_j\}$ be a sequence of finitely generated R-submodules increasing to Γ. By the Relative Monotone Theorem, it is enough to prove that $\mathcal{N} \otimes \mathcal{M}(\Gamma_j^\perp)$ is Bernoulli mod $\mathcal{N}$.

By the fundamental theorem for finitely generated modules over a principal ideal domain, Γ_j is the direct sum of a torsion R-submodule E and a free R-submodule F.

Since a submodule of a finitely generated R-module is again finitely generated, E is a finite dimensional rational vector space on which T is an isomorphism. The linear algebra of §4 using the primary and rational decompositions, which is less complicated here because we can disregard lattices, shows that T_E is a finite succession of skew products with automorphisms whose matrix is the companion matrix of some irreducible polynomial in $\mathbf{Q}[x]$. Theorem 5.1, together with another application of the Relative Monotone Theorem, show that the Skew Product Theorem holds for each of these component automorphisms. Hence we can conclude that if $K = E^\perp$, then $\mathcal{N} \otimes \mathcal{M}(K)$ is Bernoulli mod $\mathcal{N}$.

Now V is a skew product of V/K with S_K. The dual of K is the free R-module $F = Rf_1 \oplus \cdots \oplus Rf_r$. It follows that S_K is the group shift on $\hat{Q}^r$. By Theorem 3.1, the whole factor $\mathcal{N} \otimes \mathcal{M}(\Gamma_j^\perp)$ is Bernoulli mod $\mathcal{N} \otimes \mathcal{M}(K)$, which we know is Bernoulli mod $\mathcal{N}$. Hence $\mathcal{N} \otimes \mathcal{M}(\Gamma_j^\perp)$ is Bernoulli mod $\mathcal{N}$ completing the proof.

REMARK ON THE NONABELIAN CASE. It has been shown by Yuzvinskii [41], and more simply by Miles and Thomas [21], that an ergodic automorphism of a general compact group is the inverse limit of automorphisms which are a skew product of a compact abelian group automorphism with a group shift on a nonabelian compact group. Our results hold for abelian groups, and taking a further skew product with a nonabelian group shift is covered by Theorem 3.1. The Relative Monotone Theorem shows that the Skew Product Theorem is preserved under inverse limits, and hence all of our results apply to nonabelian groups as well.

8. Complemented translation invariant factors

A factor $\mathcal{A}$ of a map U of $(X, \mathcal{N}, \nu)$ is *complemented* if there is a factor $\mathcal{B}$ of U which is independent of $\mathcal{A}$ and such that $\mathcal{A} \vee \mathcal{B} = \mathcal{N}$. Such factors are of interest for they decompose the map into the direct product of two (possibly simpler) maps. Thouvenot's notion of a map being finitely determined relative to a factor gives a necessary and sufficient condition for the factor to have a Bernoulli complement, that is for $\mathcal{N}$ to be Bernoulli mod $\mathcal{A}$.

In a Bernoulli shift, there is one easily obtained necessary condition for a factor to be complemented. If $\mathcal{A}$ is such a complemented factor for U, then for any factor $\mathcal{A}'$ strictly larger than $\mathcal{A}$ we have $h(U, \mathcal{A}') > h(U, \mathcal{A})$. We shall refer to this property by saying that $\mathcal{A}$ is *entropy maximal.* It is easy to deduce from lemma 2 of [3] or from [28] that complemented factors of a Bernoulli shift are entropy maximal. However, Ornstein [26] has produced an example of an entropy maximal factor of a Bernoulli shift that is not complemented.

Certain factors of group automorphisms arise in a natural algebraic way. If S is an automorphism of the compact abelian group G with Borel sets $\mathcal{M}$ and H is a closed subgroup of G invariant under S, then $\mathcal{M}(H)$ is an "algebraic" factor of S. The following result classifies those algebraic factors that are complemented, and shows that the pathological behavior of Ornstein's example is absent for this restricted class of factors.

THEOREM 8.1. *With the notation established above, we have that the factor $\mathcal{M}(H)$ of an automorphism S is complemented if and only if the restriction automorphism S_H is ergodic if and only if $\mathcal{M}(H)$ is entropy maximal.*

PROOF. We have indicated before that if $\mathcal{M}(H)$ is complemented, then it is entropy maximal.

If H is invariant under S, then, by taking a Borel cross section to the quotient map $G \to G/H$, S can be considered as the skew product of $S_{G/H}$ with S_H, where $\mathcal{M}(H)$ is now the base factor. We consider the more general situation of a skew product $V = U \times_\phi S$ of a map U of $(X, \mathcal{N}, \nu)$ and an automorphism S of the compact abelian group H.

If S is ergodic on H, then the Splitting Theorem shows that the base algebra $\mathcal{N}$ has a Bernoulli complement, and hence is entropy maximal.

We complete the proof by showing that if S is not ergodic on H, then the base factor $\mathcal{N}$ is not entropy maximal. If S is not ergodic on H, then the dual automorphism T on $\Gamma = \hat{H}$ has nonzero periodic characters. Hence there is an integer $k > 0$ such that $P = \{\gamma \in \Gamma\colon T^k\gamma = \gamma\}$ is a nontrivial subgroup of Γ. Let $K = P^\perp$, a proper subgroup of H. Since T^k is the identity on P, S^k is the identity on H/K, so the entropy of $S_{H/K}$ is 0. The algebra $\mathcal{N} \otimes \mathcal{M}(K)$ properly contains $\mathcal{N}$ since $K \neq H$, and is a factor of V since $\mathcal{M}(K)$ is translation invariant. The skew product V on this factor is just $U \times_{\bar{\phi}} S_{H/K}$, where $\bar{\phi}$ is the image of ϕ under $H \to H/K$. By the addition theorem for entropy,

$$h(V, \mathcal{N} \otimes \mathcal{M}(K)) = h(V, \mathcal{N}),$$

proving that $\mathcal{N}$ is not entropy maximal.

9. Entropy

We shall use our look into the structure of irreducible solenoidal automorphisms from §5 to compute their entropy. The geometry developed there makes clear the meaning of each of the two terms of the formula (9.1). The entropy of such automorphisms was first calculated by Yuzvinskii (special cases were calculated earlier by Arov) and forms the central result of his paper [40] on the entropy of group endomorphisms. His arguments are algebraic, however, and are somewhat different from those here. We will conclude by discussing the set of possible values for the entropy of a group automorphism. This set turns out to be either a countable subset of $[0, \infty]$ or all of $[0, \infty]$, depending on the answer to an as yet unsettled problem first posed by D. H. Lehmer over forty years ago.

Let S be an irreducible solenoidal automorphism with matrix $C(p)$, the companion matrix of an irreducible monic polynomial $p(x)$ of degree d in $\mathbf{Q}[x]$. Let Δ be the least positive integer such that $\Delta p(x)$ has integral coefficients. Let $p(x) = \Pi_1^d(x - \lambda_i)$. A dash (as in Σ') attached to an operation indexed by the λ_i cancels our previous convention restricting the operation to $\operatorname{Im}\lambda_i \geqq 0$.

THEOREM 9.1. *With the above notation, the entropy of the irreducible solenoidal automorphism S is*

$$h(S) = \sum_{|\lambda_i|>1}' \log|\lambda_i| + \log \Delta. \tag{9.1}$$

PROOF. Bowen [6] has shown that if S is an automorphism of a compact metric group, $B(\varepsilon)$ is the ball of radius ε about the identity, and $D_n(\varepsilon, S)$ denotes $\bigcap_{k=0}^{n} S^{-k}B(\varepsilon)$, then

$$h(S) = \lim_{\varepsilon \to 0} \limsup_{n\to\infty} -\frac{1}{n}\log \mu(D_n(\varepsilon, S)). \tag{9.2}$$

Using the notation of §5, we can choose a sequence of ε's tending to zero such that $B(\varepsilon) = C \oplus (H_{-m} \cap H_q)$ where $C = C^s \oplus (\bigoplus_{|\lambda|>1} C_\lambda)$ is a mapping box of small diameter as in §5. We assume that C is small enough so that if t and u are distinct elements of the finite subgroup $\pi_H(S^{-1}H)$, then $\pi_H(C)+t$ and $\pi_H(C)+u$ are disjoint. It follows that

$$\begin{aligned} B(\varepsilon) \cap S^{-1}B(\varepsilon) &= (C \cap S^{-1}C) \oplus (H_{-m} \cap H_q \cap S^{-1}H_{-m} \cap S^{-1}H_q) \\ &= (C \cap S^{-1}C) \oplus (H_{-m-1} \cap H_q), \end{aligned}$$

and by induction that

$$D_n(\varepsilon, S) = \bigcap_{k=0}^{n} S^{-k}B(\varepsilon) = \left(\bigcap_{k=0}^{n} S^{-k}C\right) \oplus (H_{-m-n} \cap H_q). \tag{9.3}$$

To calculate the measure of $D_n(\varepsilon, S)$, first note that on $C \oplus H$ the measure μ is the product of the restriction of Lebesgue measure ω to C with Haar measure μ_H on H. We therefore must only evaluate the measure of each term on the right side of (9.3) and multiply the answers together.

Since S^{-1} multiplies distances on W_λ by $|\lambda|^{-1}$, we have $S^{-k}C^s \subset C^s$, and for $|\lambda|>1$,

$$\omega_\lambda\left(\bigcap_{k=0}^{n} S^{-j}C_\lambda\right) = \omega_\lambda(S^{-n}C_\lambda) = |\lambda|^{(-n)\dim W_\lambda}\omega_\lambda(C_\lambda).$$

Hence

$$\omega\left(\bigcap_{k=0}^{n} S^{-k}C\right) = \left(\prod_{|\lambda|>1}' |\lambda|\right)^{-n} \omega(C).$$

By the independence result of Lemma 5.2, we have $\mu_H(H_{-m-n} \cap H_q) = \mu_H(H_{-m-n})\mu_H(H_q)$. To calculate the first term, note that

$$\mu_H(H_{-m-n}) = |H/H_{-m-n}|^{-1} = \prod_{j=0}^{m+n-1} |H_{-j}/H_{-j-1}|^{-1}. \tag{9.4}$$

For $j \geqq 0$ we have

$$|S^{-j}H/H_{-j-1}| = |S^{-j}H/H_{-j}| \cdot |H_{-j}/H_{-j-1}|,$$

while, by Lemma 5.2,

$$|S^{-j}H/H_{-j-1}| = |S^{-j}H/S^{-j-1}H \cap S^{-j}H| \cdot |S^{-j}H/H_{-j}|.$$

Comparison gives

$$|H_{-j}/H_{-j-1}| = |H/H_{-1}| \qquad (j \geqq 0).$$

Now the dual of H/H_{-1} is $(\Lambda + T\Lambda)/\Lambda$, which by definition of Δ is precisely the cyclic group $\{jT^d\gamma + \Lambda : 0 \leqq j < \Delta\}$ of order Δ. Hence $|H/H_{-1}| = \Delta$. Thus by using (9.4) we find that

$$\mu_H(H_{-m-n} \cap H_q) = \mu_H(H_q)\Delta^{-m-n}.$$

Hence

$$-\frac{1}{n}\log \mu(D_n(\varepsilon, S)) = \sum_{|\lambda_i|>1}{}' \log|\lambda_i| + \frac{n+m}{n}\log\Delta + O\left(\frac{1}{n}\right),$$

and letting $n \to \infty$ in (9.1) gives the result.

We remark that, as noted by Yuzvinskii, the decomposition of solenoidal automorphisms into skew products with irreducible ones, together with the addition theorem for entropy, show that the calculation of entropy in Theorem 9.1 is valid for an arbitrary solenoidal automorphism.

Finally, we wish to investigate the possible values for the entropy of a group automorphism. The following result shows that for a group automorphism there is a "maximal" subgroup on which the automorphism is ergodic and which contains all of the entropy.

THEOREM 9.2. *Let S be an automorphism of the compact abelian group G, and T be the dual automorphism of Γ. Let $P_0 = 0$, $P_1 = \{\gamma \in \Gamma : T^k\gamma - \gamma = 0$ for some $k \neq 0\}$, and define the increasing sequence of T-invariant subgroups P_n of Γ inductively by $P_{n+1} = \{\gamma \in \Gamma : T^k\gamma - \gamma \in P_n$ for some $k \neq 0\}$. Let $P = \bigcup_0^\infty P_n$ and H be the annihilator of P. Then S_H is ergodic, and $S_{G/H}$ is the inverse limit of zero entropy automorphisms, and so also has zero entropy.*

REMARK. Since S_H is ergodic, by Theorem 8.1 it follows that $\mathcal{M}(H)$ is entropy maximal. This means that $\mathcal{M}(H)$ is exactly the Pinsker algebra of S. A slight variant of the proof below shows that $\mathcal{M}(H)$ is also the Pinsker algebra for any affine map $g \to Sg + g_0$, where g_0 is a fixed but arbitrary element of G. The fact that the Pinsker algebra of an affine transformation is an "algebraic" factor was first proved for general compact groups by Conze [9].

PROOF. If $T^k\gamma - \gamma \in P_n$, then $\gamma \in P_{n+1} \subset P$, so T is aperiodic on Γ/P, the dual of H. Hence S_H is ergodic.

Let H_n be the annihilator of P_n. The G/H is the inverse limit of the G/H_n. Since the dual of $(G/H_{n+1})/(G/H_n)$ is P_{n+1}/P_n, every element of which is periodic under T, an easy argument shows that the entropy S on $(G/H_{n+1})/(G/H_n)$ is zero. The addition theorem for entropy used inductively then gives $h(S_{G/H_n}) = 0$ for every n, and continuity of entropy under increasing limits yields $h(S_{G/H}) = 0$.

Thus in searching for the possible entropies of group automorphisms, it is enough to consider the ergodic ones. The proof in §6 concerning ergodic automorphisms of totally disconnected groups shows that they always have a group shift as a factor, hence have entropy $\geqq \log 2$. Our evaluation of the entropy of a solenoidal automorphism shows that it is bounded below by $\log \Delta \geqq \log 2$ unless $\Delta = 1$, i.e. unless it is a toral automorphism. Thus the only possibilities for ergodic group automorphisms with small entropy are those of a torus. This leads directly to the following.

LEHMER'S PROBLEM. If $p(x) = \Pi_i(x - \lambda_i)$ is a monic polynomial with integral coefficients and constant term ± 1, can $\Sigma_{|\lambda_i|>1} \log |\lambda_i|$ be arbitrarily small?

Lehmer asked this question over forty years ago [16], and the answer is still unknown. In the same paper he found the smallest value known to date, namely log 1.176280821, which corresponds to the polynomial

$$p(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1 .$$

C. L. Siegel [32] showed that if just one of the λ_i is on or outside the unit circle (that is, this root is a Pisot–Vijayaraghavan number), then the logarithm of the positive root of $x^3 - x - 1$ (about log 1.324) is the smallest possible. P. E. Blanksby and H. L. Montgomery [4] have proved that for polynomials of degree d,

$$\sum_{|\lambda_i|>1} \log |\lambda_i| \geqq \log\left(1 + \frac{1}{52d \log d}\right).$$

If the answer to Lehmer's problem is "yes", so that there are ergodic toral automorphisms with arbitrarily small entropy, then clearly by taking direct products of possibly a countable number of these, any positive entropy for an automorphism of $\mathbf{T}^\infty$ can be achieved. Conversely, we showed in a previous paper [18] that the existence of an automorphism of $\mathbf{T}^\infty$ with finite positive entropy implies that there are toral automorphisms of arbitrarily small entropy.

If, however, the answer to Lehmer's problem is "no", then from the proof in §7 for building up a general ergodic group automorphism from skew products with solenoids and group shifts it follows from the addition formula that the set of possible values for the entropy of such an automorphism is only countable. Thus we have established our last result.

THEOREM 9.3. *The set of possible values for the entropy of a group automorphism is either a countable subset of* $[0,\infty]$ *or all of* $[0,\infty]$, *depending on the answer to Lehmer's problem. Also, the group* $\mathbf{T}^\infty$ *either has no automorphisms of finite positive entropy, or automorphisms of every finite positive entropy, again depending on the answer to Lehmer's problem.*

Appendix: Relative ergodic theory with infinite entropy

We have made essential use in this paper of the Relative Factor and Relative Monotone Theorems. Thouvenot's proofs of these results in [36] apply only to factors with finite entropy, since he deals only with factors generated by a finite partition. We indicate here how these results and their proofs can be carried over to factors with infinite entropy. Briefly, using the fact that every factor of an ergodic map has a countable generator, conditioning factors with infinite entropy are handled by using a countable instead of a finite generator, replacing certaining entropy statements that then become indeterminate by well-defined relative entropy statements, and using a relative version of the Shannon–McMillan theorem. Factors of infinite entropy that are finitely determined (i.e. Bernoulli) relative to a conditioning factor (of possibly infinite entropy) are then handled just as in the absolute case.

We assume that the reader is familiar with Thouvenot's paper [36], and Ornstein's treatment of infinite entropy [27, Part I, §9]. For ease of comparison, we use Thouvenot's notation, although it is different from that employed previously here. The only departure from this is our use of h instead of E to denote entropy. Thus if P is a partition, $d(P)$ denotes the distribution of P, $d(P,P')$ and $|P-P'|$ are the distributional and partition distances between P and P', and $h(T,P)$ is the entropy of T on P.

If T is a map of X, P a finite partition of X, and H a countable partition of X that generates a factor $\mathcal{H} = \bigvee_{-\infty}^{\infty} T^iH$, define the conditional entropy of T on P mod $\mathcal{H}$ to be

$$h(T,P/\mathcal{H}) = h(P \mid \bigvee_{1}^{\infty} T^{-i}P \vee \mathcal{H}).$$

The entropy of T mod $\mathcal{H}$ is

$$h(T/\mathcal{H}) = \sup_P h(T,P/\mathcal{H}),$$

where the supremum is over all finite partitions of X.

Let P be finite and H be countable. Then P is called *H-conditionally finitely determined* if for every $\varepsilon > 0$ there is a $\delta > 0$ and an integer n such that if $\bar{T}$ is an ergodic map of $\bar{X}$ and there are partitions $\bar{P}$ and $\bar{H}$ of $\bar{X}$ such that for $\bar{\mathscr{H}} = \bigvee_{-\infty}^{\infty} \bar{T}^i \bar{H}$, we have

$$d\left(\bigvee_0^m \bar{T}^i \bar{H}\right) = d\left(\bigvee_0^m T^i H\right) \qquad (m \geqq 0),$$

$$d\left(\bigvee_0^n \bar{T}^i (\bar{P} \vee \bar{H}),\ \bigvee_0^n T^i (P \vee H)\right) < \delta,$$

and

$$(*) \qquad | h(\bar{T}, \bar{P}/\bar{\mathscr{H}}) - h(T, P/\mathscr{H}) | < \delta,$$

then there is a space Z, and for every positive integer p sequences $\{H_i\}_0^p$, $\{P_i\}_0^p$, $\{\bar{P}_i\}_0^p$ of partitions of Z such that

$$d\left(\bigvee_0^p T^i (P \vee H)\right) = d\left(\bigvee_0^p P_i \vee H_i\right),$$

$$d\left(\bigvee_0^p \bar{T}^i (\bar{P} \vee \bar{H})\right) = d\left(\bigvee_0^p \bar{P}_i \vee H_i\right),$$

and

$$| P_i - \bar{P}_i | < \varepsilon \qquad (0 \leqq i \leqq p).$$

This definition coincides with Thouvenot's for finite H, the only difference being that we have replaced the possibly indeterminate expression $| h(\bar{P} \vee \bar{H}, T) - h(P \vee H, T) |$ by $(*)$.

When phrasing conditions in terms of relative entropies, the only additional result required is a relative Shannon–McMillan theorem, which we will now state and sketch a proof of. If $P = \{P_1, \cdots, P_k\}$ is a partition of X, and $\mathscr{A}$ is a subalgebra, the conditional information of P given $\mathscr{A}$ is the function

$$I(P \mid \mathscr{A}) = \sum_{i=1}^{k} (-\log E(\chi_{P_i} \mid \mathscr{A}))\chi_{P_i},$$

where $E(\chi_{P_i} \mid \mathscr{A})$ is the conditional expectation of the characteristic function of P_i with respect to $\mathscr{A}$. Then one has

$$h(T, P/\mathscr{H}) = \int_X I\left(P \ \bigvee_1^\infty T^{-i} P \vee \mathscr{H}\right) d\nu.$$

The most convenient form of the result we want is the following.

Relative Shannon–McMillan Theorem. *Let P be a finite partition and H be a countable partition of X, T be an ergodic map of X, and $\mathscr{H} = \bigvee_{-\infty}^{\infty} T^i H$. Then*

$$\frac{1}{n} I\left(\bigvee_0^{n-1} T^i P \mid \bigvee_0^{n-1} T^i H\right) \rightarrow h(T, P/\mathscr{H})$$

in $L^1(X)$.

PROOF. This follows as in the standard proof (see [30]) by writing the left side as

$$\frac{1}{n} \sum_{k=0}^{n-1} T^k I\left(P \mid \bigvee_{-k}^{-1} T^i P \vee \bigvee_{-k}^{n-k-1} T^i H\right),$$

noticing that the martingale convergence theorem together with an integrable dominating function for conditional entropies (lemma 2.1 of [30]) show that

$$I\left(P \mid \bigvee_{-k}^{-1} T^i P \vee \bigvee_{-r}^{s} T^i H\right) \rightarrow I(T, P/\mathscr{H})$$

in $L^1(X)$ as $k, r, s \rightarrow \infty$, and invoking the L^1 ergodic theorem.

Using this result together with the appropriate changes, Thouvenot's results and proofs go through for countable H. For example, his relative Sinai theorem (Proposition 2) takes the following form.

PROPOSITION 2. *Let T be an ergodic map of X and H a countable partition of X generating the factor $\mathscr{H}$. Suppose that $h(T/\mathscr{H}) < \infty$, and let I be a finite probability distribution whose entropy is $h(T/\mathscr{H})$. Then given $\varepsilon > 0$, there is a $\delta > 0$ such that if P' is a partition of X with $d(P', I) < \delta$ and $0 < h(T/\mathscr{H}) - h(T, P/\mathscr{H}) < \delta$, then there is a partition P of X such that $|P - P'| < \varepsilon$, $d(P) = I$, $\{T^i P \colon i \in \mathbf{Z}\}$ is independent, and $\bigvee_{-\infty}^{\infty} T^i P \perp \mathscr{H}$.*

Thus the Factor and Monotone Theorems extend the case when the conditioning factor $\mathscr{H}$ has infinite entropy. We now discuss the case of an infinite entropy factor being finitely determined mod $\mathscr{H}$.

If $\mathscr{A}$ is a factor of T, we say $\mathscr{A}$ is an increasing limit of $\mathscr{H}$-conditionally finitely determined factors if there is an increasing sequence of factors $\mathscr{A}_n$ with $h(T, \mathscr{A}_n/\mathscr{H}) < \infty$ and each $\mathscr{A}_n$ is $\mathscr{H}$-conditionally finitely determined. By using the relative Sinai theorem (Proposition 2) together with the fact that a partition that generates a Bernoulli complement of relative full entropy can be modified by an arbitrarily small amount to yield a relative Bernoulli generator (proposition 3 of [36]), one shows that $\mathscr{A}$ is the increasing limit of $\mathscr{H}$-conditionally finitely determined factors if and only if $\mathscr{A}$ is Bernoulli mod $\mathscr{H}$, the Bernoulli complement having, of course, entropy $h(T, \mathscr{A}/\mathscr{H})$. This practically finishes the proof of the Relative Monotone Theorem for infinite entropy. Suppose the factors $\mathscr{A}_n$ increase to $\mathscr{A}$, and that each $\mathscr{A}_n$ is Bernoulli mod $\mathscr{H}$. Since $\mathscr{A}$ is generated by a countable partition, we can assume that each $\mathscr{A}_n$ is generated mod $\mathscr{H}$ by a finite partition, that is, $\mathscr{A}_n \vee \mathscr{H} = \bigvee_{-\infty}^{\infty} T^i P_n \vee \mathscr{H}$ for a finite P_n. A sequence of applications of propositions 2 and 3 of [36], as in the absolute case, shows that $\mathscr{H}$ is Bernoulli complemented in $\mathscr{A}$.

Finally, suppose that a factor $\mathscr{A}$ is Bernoulli mod $\mathscr{H}$, and that $\mathscr{B}$ is a σ-subalgebra of $\mathscr{A}$. First note that by the Relative Monotone Theorem, we can assume that $\mathscr{B}$ is generated mod $\mathscr{H}$ by a finite partition P. If $h(T, \mathscr{A}/\mathscr{H}) < \infty$, this is proposition 4 of [36]. If $h(T, \mathscr{A}/\mathscr{H}) = \infty$, there is a sequence $\mathscr{A}_n \nearrow \mathscr{A}$ with $h(T, \mathscr{A}_n/\mathscr{H}) < \infty$ and $\mathscr{A}_n$ Bernoulli mod $\mathscr{H}$. There are $\mathscr{A}_n$-measurable partitions P_n such that $|P_n - P| \to 0$. By the Relative Factor Theorem for finite entropy, each P_n is $\mathscr{H}$-conditionally finitely determined. Now the relative $\bar{d}$ distance $\bar{d}_H(P, P_n) \to 0$, and the relative $\bar{d}$ limit of the $\mathscr{H}$-conditionally finitely determined partitions is again $\mathscr{H}$-conditionally finitely determined (proposition 7 of [36]). Thus $\mathscr{A}$ is Bernoulli mod $\mathscr{H}$, completing the proof of the general Relative Factor Theorem.

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