

## Products of Coboundaries for Commuting Nonsingular Automorphisms

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Let  $U$  and  $V$  be commuting nonsingular automorphisms of the Lebesgue measure space  $(X, \mu)$ , that is, both are invertible, measurable, and map  $\mu$ -null sets to  $\mu$ -null sets. Assume that the fixed point set of  $U^m V^n$  has measure zero unless  $m=n=0$ . While developing a cohomology theory for the orbit equivalence relation of a nonsingular action of a countable group on  $X$ , Feldman and Moore ([4, Theorem 4], [5, Theorem 7]) proved the following result. If  $G$  is an abelian polonais group, and  $h: X \rightarrow G$  is measurable, then there are measurable  $f, g: X \rightarrow G$  such that

$$h(x) = \frac{f(Ux)}{f(x)} \frac{g(Vx)}{g(x)} \quad \text{a.e.} \quad (1)$$

Their proof uses the hyperfiniteness of the nonsingular  $\mathbb{Z}^2$  action generated by  $U$  and  $V$  to show that a certain two-dimensional cohomology group vanishes, establishes that this group is isomorphic to a two-dimensional Eilenberg-MacLane cohomology group of  $\mathbb{Z}^2$  with certain coefficients, and then uses spectral sequence methods to calculate the consequences.

They remark that even for  $U$  and  $V$  independent irrational rotations of the circle group, and  $G$  the circle group or the reals, “we obtain very concrete and elementary statements that seem quite inaccessible by any other method.”

My purpose here is to give a direct proof of their result, using only the Halmos-Rohlin theorem for nonsingular  $\mathbb{Z}^2$  actions. The latter theorem seems to be a key ingredient, for it quickly yields the hyperfiniteness of nonsingular  $\mathbb{Z}^2$  actions which begins the Feldman-Moore proof.

The factorization (1) can be regarded as a functional equation in the two variables  $f$  and  $g$ . The Halmos-Rohlin theorem is useful in solving other functional equations as well.

For example, a complex number  $\lambda$  is an eigenvalue for a measure-preserving transformation  $U$  of  $X$  precisely when the functional equation

$$\lambda = \frac{f(Ux)}{f(x)}$$

can be solved for measurable  $f$  mapping  $X$  to the circle group. An argument using the Halmos-Rohlin theorem gives a quick proof (unpublished) of Hansel's result [6] that an ergodic transformation induces an arbitrary eigenvalue on a dense collection of subsets (Conze [1] had previously proved this for a special class of eigenvalues).

A functional equation also arises in the splitting of certain skew products. Let  $U$  be a measure-preserving transformation of  $X$ ,  $A$  be an automorphism of a compact abelian group  $G$ , written multiplicatively, and  $h: X \rightarrow G$  be measurable. Define the skew product  $U \times_h A: X \times G \rightarrow X \times G$  by  $(U \times_h A)(x, t) = (Ux, h(x)(At))$ . This skew product is then isomorphic to the direct product  $U \times A$  via an isomorphism of the form  $(x, t) \mapsto (x, f(x)t)$ , where  $f: X \rightarrow G$ , if and only if  $f$  is a solution of the functional equation

$$h(x) = \frac{f(Ux)}{Af(x)}. \tag{2}$$

I have shown in [9] how to use the Halmos-Rohlin theorem, together with a property of homeomorphisms called specification, to prove that for every transformation  $U$ , every ergodic automorphism  $A$ , and every  $h$ , there is a solution  $f$  of (2). One consequence is a simpler proof of the Bernoullicity of ergodic toral automorphisms (see [8, §4]).

I will now state the result I will prove.

**Theorem.** *If  $U$  and  $V$  are commuting nonsingular automorphisms of  $(X, \mu)$  such that  $U^m V^n$  has fixed point set of measure zero unless  $m = n = 0$ , and if  $G$  is a group, then for every  $h: X \rightarrow G$  there are  $f, g: X \rightarrow G$  such that*

$$h(x) = f(Ux) f(x)^{-1} g(Vx) g(x)^{-1} \quad \text{a.e.} \tag{3}$$

Some remarks before beginning the proof. If  $G$  comes equipped with a  $\sigma$ -subalgebra of measurable sets (in [4, 5]  $G$  is polonais and therefore has a natural Borel structure), and  $h$  is measurable, the construction below easily gives measurable  $f$  and  $g$ . Also, using the axiom of choice to pick out one point in each orbit of  $V$  in the exceptional null set of (3), and redefining  $g$  inductively using (3) starting from its values at these points, it follows that  $f$  and  $g$  can be chosen so that (3) holds everywhere. Finally, this proof allows  $G$  to be non-abelian.

*Proof.* First some notation. Since  $U$  and  $V$  commute, they generate a nonsingular action of  $\mathbb{Z}^2$  on  $X$  given by  $(m, n) \cdot x = U^m V^n x$ , where  $x \in X$  and  $(m, n) \in \mathbb{Z}^2$ . If  $B \subset \mathbb{Z}^2$  and  $F \subset X$ , then  $BF$  denotes  $\bigcup_{b \in B} bF$ . Also,  $F$  is called a  $B$ -set if  $\{bF: b \in B\}$  is a disjoint collection. If  $A, B \subset \mathbb{Z}^2$ , then  $\bigcap_A B$  denotes  $\bigcap_{a \in A} (B - a)$ . Our assumptions on  $U$  and  $V$  mean that the resulting  $\mathbb{Z}^2$  action is aperiodic, that is,  $\mu\{x: ax = x\} = 0$  for nonzero  $a$  in  $\mathbb{Z}^2$ . The following special case of Theorem 1 in [3], in which  $\mathbb{Z}^2$  here replaces a general countable abelian group, is the proof's foundation.

**Halmos-Rohlin Theorem.** *If  $\mathbb{Z}^2$  acts aperiodically and nonsingularly on  $(X, \mu)$ , then for every subset  $A$  of  $\mathbb{Z}^2$  and every  $\varepsilon > 0$ , there is a square  $B$  in  $\mathbb{Z}^2$  and a measurable  $B$ -set  $F$  in  $X$  with  $\mu[(\bigcap_A B)F] > 1 - \varepsilon$ .*

Conze [2] and Katznelson and Weiss [7] proved this in the measure-preserving case, and this is all that is needed for the application to independent rotations mentioned above. Nonsingular actions are trickier because sets which are easily shown to have small measure in the measure-preserving case must be dealt with by averaging arguments in [3].

The proof uses the Halmos-Rohlin theorem to construct  $f$  and  $g$  inductively on an increasing sequence of subsets so that (3) will hold on each subset. On the union of these sets, which will have measure 1,  $f$  and  $g$  will be defined and obey (3).

To continue the proof, let  $A = \{0, 1\}^2$ . Using the Halmos-Rohlin theorem, inductively find squares  $B_k = \{0, 1, \dots, n_k - 1\}^2 = I_k^2$  and  $(B_k + A)$ -sets  $\bar{F}_k$  with  $\mu[\bigcap_{B_{k-1}} B_k \bar{F}_k] > 1 - 2^{-k}$ . If  $F_k = \bar{F}_k \cap [\bigcap_{j>k} (\bigcap_{B_j} B_{j+1}) \bar{F}_{j+1}]$ , then  $F_k$  is again a  $B_k$ -set,  $B_k F_k$  increases,  $\mu(B_k F_k) > 1 - (2^{-k} + 2^{-k-1} + \dots) = 1 - 2^{-k+1}$ , and for  $j > k$  and  $x \in F_j$ ,  $B_k F_k$  intersects  $B_j x$  in complete squares of side  $n_k$  sprinkled about in  $B_j$ .

Let  $\bar{I}_k = \{0, 1, \dots, n_k\} = I_k \cup \{n_k\}$ ,  $B'_k = \bar{I}_k \times I_k$ ,  $B''_k = I_k \times \bar{I}_k$ ,  $E_k = B_k F_k$ ,  $E'_k = B'_k F_k = E_k \cup U E_k$ ,  $E''_k = B''_k F_k = E_k \cup V E_k$ . Thus  $B'_k$  is the square  $B_k$  with a border of thickness one on the right, and  $B''_k$  is  $B_k$  with a border on top. Both  $E'_k$  and  $E''_k$  increase to sets of full measure, and I will inductively define  $f$  on each  $E''_k$  and  $g$  on each  $E'_k$  so that (3) holds on  $E_k = E'_k \cap E''_k$ . Since  $E_k$  also increases to a set of full measure, the theorem will be proved.

Define  $f$  arbitrarily on  $E''_1$ , measurably if  $G$  is measurable. Define  $g$  arbitrarily on  $(I_1 \times \{0\})F_1$ , measurably if  $G$  is measurable, and then up the columns of  $B'_1$  using (3) as follows. If  $x \in (I_1 \times \{0\})F_1$ , put

$$g(Vx) = [f(x) f(Ux)^{-1} h(x)] g(x),$$

and continue inductively to get

$$\begin{aligned} g(V^{j+1}x) &= [f(V^jx) f(UV^jx)^{-1} h(V^jx)] g(V^jx) \\ &= \left[ \prod_{m=0}^j f(V^m x) f(UV^m x)^{-1} h(V^m x) \right] g(x) \end{aligned}$$

for  $0 \leq j < n_1$ , where  $\prod_{m=0}^j g_m$  means  $g_j g_{j-1} \dots g_0$ . This defines  $g$  on

$$\bigcup_{j=0}^{n_k} V^j((I_1 \times \{0\})F_1) = (I_1 \times \bar{I}_1)F_1 = E''_1,$$

and (3) holds for  $x \in E_1$ .

Now assume  $f$  and  $g$  have been defined on  $E''_{k-1}$  and  $E'_{k-1}$ , respectively, and that (3) holds for  $x \in E_{k-1}$ . Notice that for  $x \in F_k$ , the set  $E''_{k-1}$  occurs in  $B_k x$  as rectangles, each of size  $n_{k-1} \times (n_{k-1} + 1)$ , which I call  $(k-1)$ -rectangles. I will extend the definition of  $g$  from these  $(k-1)$ -rectangles where it is already defined to all of  $B_k x$ . This extension is done one column at a time, using some flexibility in defining  $f$  on the next column to the right.

Let  $C = \{0\} \times I_k$ ,  $\bar{C} = \{0\} \times \bar{I}_k$ , and fix  $x \in F_k$ . Extend  $f$  arbitrarily from  $Cx \cap E''_{k-1}$  to  $Cx$ . Let  $J = \{j: 0 \leq j \leq n_k, V^j x \in (I_{k-1} \times \{0\})F_{k-1}\}$ , so  $J$  specifies the

times when the first column intersects the base of a  $(k-1)$ -rectangle. Since  $F_{k-1}$  is a  $(B_{k-1} + A)$ -set, the  $(k-1)$ -rectangles are spaced at least a unit distance apart. Hence for  $j \in J$ , it follows that  $(1, j-1)x = UV^{j-1}x \notin E_{k-1}^r$ , and therefore  $f$  is not defined at such points. Let  $C' = \{1\} \times I_k$ , the next column to the right of  $C$ , and extend  $f$  arbitrarily from  $C'x \cap E_{k-1}^r$  to  $\{UV^i x: 0 \leq i < n_k, i+1 \notin J\}$ . Extend  $g$  to  $Cx$  using  $f$  as follows. If  $p$  and  $q$  are consecutive elements of  $J$ , then  $q-p \geq n_{k-1} + 1$ , and  $g$  is already defined on

$$\{V^{p+i}x: 0 \leq i \leq n_{k-1}\} \cup \{V^{q+i}x: 0 \leq i \leq n_{k-1}\},$$

but not on  $\{V^i x: p+n_{k-1}+1 \leq i \leq q-1\}$ . Use (3) to define  $g$  across this gap by setting

$$g(V^i x) = \left[ \prod_{m=p+n_{k-1}+1}^i f(V^m x) f(UV^m x)^{-1} h(V^m x) \right] g(V^{p+n_{k-1}} x) \tag{4}$$

for  $p+n_{k-1}+1 \leq i \leq q-1$ . The problem now is to fit together the just defined  $g(V^{q-1}x)$  and the previously defined  $g(V^q x)$  to make (3) work. The solution is simply to rig the value of  $f$  at  $UV^{q-1}x$ , where it is not yet defined, to be

$$f(UV^{q-1}x) = h(V^{q-1}x) g(V^{q-1}x) g(V^q x)^{-1} f(V^{q-1}x).$$

Let  $r = \min J$ . This process defines  $f$  on all of  $C'x$ , except at  $UV^{r-1}x$  if  $r > 0$ , and there let  $f$  be arbitrary. If  $r > 0$ , define  $g$  on  $\{V^i x: 0 \leq i < r\}$  by

$$g(V^{r-i}x) = \left[ \prod_{m=1}^i h(V^{r-m}x)^{-1} f(UV^{r-m}x) f(V^{r-m}x)^{-1} \right] g(V^r x),$$

where  $1 \leq i \leq r$ . To finish the first column, if  $s = \max J$ , define  $g$  on  $\{V^i x: s+n_{k-1}+1 \leq i \leq n_k\}$  by Equation (4) with  $p$  replaced by  $s$ . Thus  $g$  is defined on  $\bar{C}x = (\{0\} \times \bar{I}_k)x$ ,  $f$  is defined on  $(C \cup C')x = (\{0, 1\} \times I_k)x$ , and (3) holds on  $Cx$ .

Proceeding inductively, if  $g$  is defined on  $(\{0, 1, \dots, j-1\} \times \bar{I}_k)x$  and  $f$  on  $(\{0, 1, \dots, j\} \times I_k)x$  so that (3) holds on  $(\{0, 1, \dots, j-1\} \times I_k)x$ , exactly the same procedure employed for the first column extends the definition of  $g$  to  $(\{0, 1, \dots, j\} \times \bar{I}_k)x$  and of  $f$  to  $(\{0, 1, \dots, j+1\} \times I_k)x$ , with (3) holding on  $(\{0, 1, \dots, j\} \times I_k)x$ . Continuing to  $j = n_k - 1$  provides the required extensions of  $f$  to  $E_k^r$  and of  $g$  to  $E_k^l$  with (3) holding on  $E_k$ . This completes the proof.

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