Math 444/445 **Geometry for Teachers** **Summer 2008**

Supplement C: Similar Triangles

This supplement is meant to be read after Venema's Section 9.2. Throughout this section, we assume all nine axioms of Euclidean geometry.

Similar Triangles

The idea of *scaling* geometric objects is ubiquitous in our experience. When you draw a map to scale, or enlarge a photo, or tell your computer to use a larger font size, you are creating a new geometric object that has the "same shape" as the old one, but has all of its parts reduced or enlarged in size – or "scaled" – by the same ratio. In geometry, two figures that have the same shape but not necessarily the same size are said to be *similar* to each other. (We will give a more precise mathematical definition below.)

To analyze this concept in the context of axiomatic Euclidean geometry, let us start with triangles, the simplest geometric figures. There are two separate things that we might expect to be the case when two triangles are similar (Fig. 1): First, all three pairs of corresponding angles should be congruent (which means

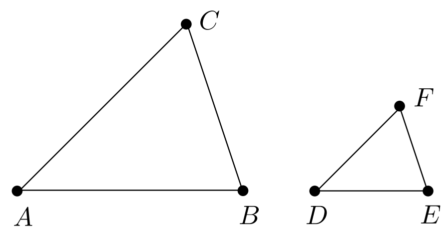

Figure 1: Similar Triangles.

the triangles have the "same shape"), and second, the lengths of pairs of corresponding sides should all have the same ratio (which means they have "proportional sizes"). In some high-school geometry texts, including that of Jacobs, the definition of similar triangles includes both of these properties. However, it is a fact (which we will prove below) that each of these conditions implies the other. For us, it will be easier to choose one of them as our official definition of similar triangles.

Thus we make the following definition: Two triangles are said to be ***similar*** if there is a correspondence between their vertices such that corresponding angles are congruent. The notation $\triangle ABC \sim \triangle DEF$ means that $\triangle ABC$ is similar to $\triangle DEF$ under the correspondence $A \leftrightarrow D$, $B \leftrightarrow E$, and $C \leftrightarrow F$, or more specifically that

$$\angle A \cong \angle D, \qquad \angle B \cong \angle E, \qquad \angle C \cong \angle F.$$

The first thing to notice is that in Euclidean geometry, it is only necessary to check that *two* of the corresponding angles are congruent.

Theorem C.1 (AA Similarity Theorem). *If $\triangle ABC$ and $\triangle DEF$ are triangles such that $\angle A \cong \angle D$ and $\angle B \cong \angle E$, then $\triangle ABC \sim \triangle DEF$.*

Proof. Under these hypotheses, it follows immediately from the Angle-Sum Theorem that $\angle C \cong \angle F$. □

The next theorem shows that similar triangles can be readily constructed in Euclidean geometry, once a new size is chosen for one of the sides. It is an analogue for similar triangles of Venema's Theorem 6.2.4.

Theorem C.2 (Similar Triangle Construction Theorem). *If $\triangle ABC$ is a triangle, $\overline{DE}$ is a segment, and H is a half-plane bounded by $\overleftrightarrow{DE}$, then there is a unique point $F \in H$ such that $\triangle ABC \sim \triangle DEF$.*

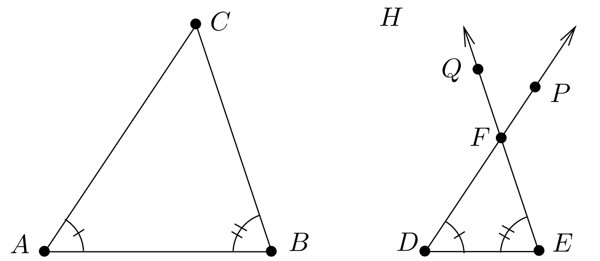

Figure 2: Construction of a triangle similar to a given one.

Proof. The Angle Construction Theorem ensures that there exists a unique ray $\overrightarrow{DP}$ lying in H such that $\angle EDP \cong \angle A$ (Fig. 2). Similarly, there exists a unique ray $\overrightarrow{EQ}$ in H such that $\angle DEQ \cong \angle B$. Because the measures of $\angle A$ and $\angle B$ sum to less than $180°$ by Corollary B.9, it follows that the measures of $\angle EDP$ and $\angle DEQ$ also sum to less than $180°$. By Euclid's Postulate V, therefore, there is a point F on that same side of $\overleftrightarrow{DE}$ where the lines $\overleftrightarrow{DP}$ and $\overleftrightarrow{EQ}$ intersect. By the AA Similarity Theorem, $\triangle ABC \sim \triangle DEF$.

To prove uniqueness, just note that if F' is any other point satisfying the conclusion of the theorem, then the fact that $\angle EDF' \cong \angle BAC$ implies that F' must lie on $\overrightarrow{DP}$ by the uniqueness part of the Angle Construction Theorem, and similarly F' must lie on $\overrightarrow{EQ}$. Since these two rays intersect only at F, it follows that $F = F'$. □

It is important to observe that although it is always possible in neutral geometry to construct a triangle *congruent* to a given one, as Theorem 6.2.4 showed, this construction of *similar* triangles only works in Euclidean geometry because it requires Euclid's Postulate V. In fact, we will see later that in hyperbolic geometry, it is impossible to construct non-congruent similar triangles!

Our main order of business in this section is to show that similar triangles have proportional sides. Our proof of this fact is modeled on that of Euclid (which he carries out in Book VI), using the theory of area. Thus before we get to our theorem about similar triangles, let us establish two simple facts about areas of triangles. The first of these is Euclid's Proposition I.37, while the second is his first proposition in Book VI. For us, they are both simple consequences of the area formula for triangles.

Lemma C.3. *Suppose $\triangle ABC$ and $\triangle A'BC$ are two distinct triangles that have a common side $\overline{BC}$, such that $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BC}$ (Fig. 3). Then $\alpha(\triangle ABC) = \alpha(\triangle A'BC)$.*

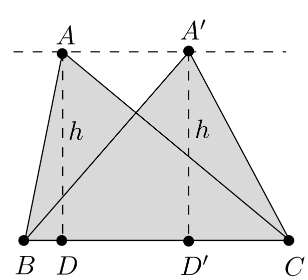

Figure 3: Lemma C.3.

Proof. Let D and D' be the feet of the perpendiculars from A and A', respectively, to $\overleftrightarrow{BC}$. Because $\overleftrightarrow{AD}$ and $\overleftrightarrow{A'D'}$ are both perpendicular to $\overleftrightarrow{BC}$, they are parallel to each other by Corollary 6.5.8. It then follows that $\square AA'D'D$ is a parallelogram (actually a rectangle, but all we need to know is that it is a parallelogram), and so $\overline{AD} \cong \overline{A'D'}$ by Theorem B.19. Set $h = AD = A'D'$, so that h is the height of both triangles $\triangle ABC$ and $\triangle A'BC$. It then follows from the area formula for triangles that $\alpha(\triangle ABC) = \frac{1}{2}(BC)h = \alpha(\triangle A'BC)$. □

Lemma C.4. *Suppose $\triangle ABC$ is a triangle, and D is a point such that $B * D * C$ (Fig. 4). Then*

$$\frac{\alpha(\triangle ABD)}{\alpha(\triangle ABC)} = \frac{BD}{BC}.$$

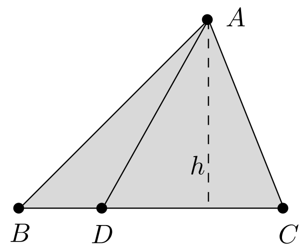

Figure 4: Lemma C.4.

Proof. Note that both $\triangle ABD$ and $\triangle ABC$ have the same height h, which is just the distance from A to $\overleftrightarrow{BC}$. Therefore, the formula for the area of a triangle shows that

$$\frac{\alpha(\triangle ABD)}{\alpha(\triangle ABC)} = \frac{\frac{1}{2}BD \cdot h}{\frac{1}{2}BC \cdot h} = \frac{BD}{BC},$$

as claimed. □

Our main tool for analyzing proportionality in similar triangles will be the following theorem, which shows that a line parallel to one side of a triangle cuts off proportional segments from the other two sides.

Theorem C.5 (The Side-Splitter Theorem). *Suppose $\triangle ABC$ is a triangle, and ℓ is a line parallel to $\overleftrightarrow{BC}$ that intersects $\overline{AB}$ at an interior point D (Fig. 5). Then ℓ also intersects $\overline{AC}$ at an interior point E, and*

$$\frac{AD}{AB} = \frac{AE}{AC}.$$

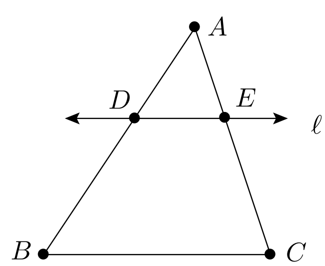

Figure 5: The Side-Splitter Theorem.

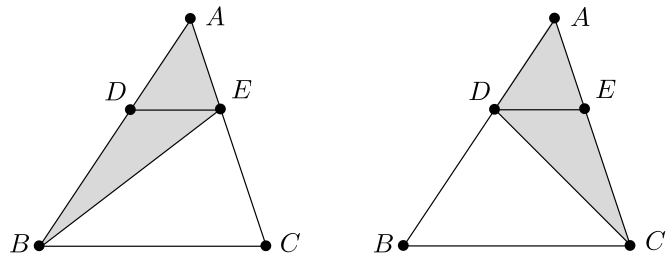

Figure 6: Proof of the Side-Splitter Theorem.

Proof. Because ℓ is parallel to $\overleftrightarrow{BC}$, it does not contain B or C; and because it intersects $\overline{AB}$ at an interior point, it does not contain A either. Therefore, Pasch's Theorem guarantees that ℓ also intersects the interior of one other side of $\triangle ABC$. Since it cannot intersect $\overline{BC}$, it must intersect the interior of $\overline{AC}$. Let E denote the intersection point.

Draw $\overline{BE}$, and consider $\triangle AEB$ (Fig. 6). From Lemma C.4, we conclude that

$$\frac{\alpha(\triangle ADE)}{\alpha(\triangle ABE)} = \frac{AD}{AB}. \tag{C.1}$$

Similarly, drawing $\overline{DC}$ and considering $\triangle ADC$, we obtain

$$\frac{\alpha(\triangle ADE)}{\alpha(\triangle ADC)} = \frac{AE}{AC}. \tag{C.2}$$

It follows from Theorem 9.1.7 and additivity of area that

$$\alpha(\triangle ABE) = \alpha(\triangle ADE) + \alpha(\triangle DEB),$$
$$\alpha(\triangle ADC) = \alpha(\triangle ADE) + \alpha(\triangle DEC).$$

But it follows from Lemma C.3 that $\alpha(\triangle DEB) = \alpha(\triangle DEC)$, and therefore that

$$\alpha(\triangle ABE) = \alpha(\triangle ADC). \tag{C.3}$$

Combining (C.1), (C.2), and (C.3), we obtain

$$\frac{AD}{AB} = \frac{\alpha(\triangle ADE)}{\alpha(\triangle ABE)} = \frac{\alpha(\triangle ADE)}{\alpha(\triangle ADC)} = \frac{AE}{AC}$$

as desired. □

Now we come to the main theorem of this section, which says that similar triangles have proportional corresponding sides. It is often called simply the ***Similar Triangles Theorem***.

Theorem C.6 (Fundamental Theorem on Similar Triangles). *If* $\triangle ABC \sim \triangle DEF$*, then*

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}. \tag{C.4}$$

Proof. Suppose $\triangle ABC \sim \triangle DEF$. If $AB = DE$, then $\triangle ABC$ is congruent to $\triangle DEF$ by *SAS*, and the theorem is true because all the ratios in (C.4) are equal to 1. So let us suppose that $AB \neq DE$. One of them is larger, say $DE > AB$. We will prove the first equality in (C.4); the proof of the other equality is exactly the same.

Choose a point P in the interior of $\overline{DE}$ such that $\overline{DP} \cong \overline{AB}$, and let ℓ be the line through P and parallel to $\overleftrightarrow{EF}$ (Fig. 7). It follows from the Side-Splitter Theorem that ℓ intersects $\overline{DF}$ at an interior point Q, and

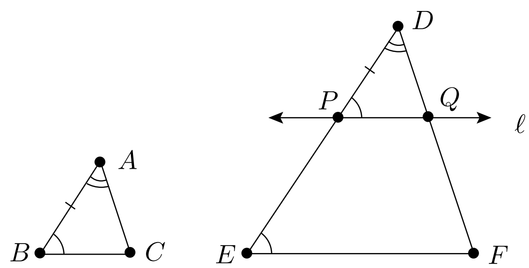

Figure 7: Proof of the Similar Triangles Theorem.

that

$$\frac{DP}{DE} = \frac{DQ}{DF}. \tag{C.5}$$

By the converse to the Corresponding Angles Theorem, $\angle DPQ \cong \angle E$, which by hypothesis is congruent in turn to $\angle B$. It also follows from the hypothesis that $\angle D \cong \angle A$. Since $\overline{DP} \cong \overline{AB}$ by construction, we have $\triangle DPQ \cong \triangle ABC$ by SAS. Substituting $DP = AB$ and $DQ = AC$ into (C.5), we obtain the first equation in (C.4). □

This is one of the most important theorems in Euclidean geometry. Nearly every geometry book has some version of it, but you will find it treated differently in different books. For example, in a textbook that includes proportionality of the sides as part of the *definition* of similar triangles, this theorem would be rephrased to say that if two triangles have congruent corresponding angles, then they are similar. Some textbooks, for some reason, do not prove this theorem at all, but instead take it as an additional postulate (often called the *AAA Similarity Postulate* or some such thing).

We have followed the lead of Euclid (as does Jacobs) in using the theory of area to prove this theorem. In fact, it is possible to give a proof that does not use areas at all (and therefore does not require either the Neutral Area Axiom or the Euclidean Area Axiom). Venema gives such a proof in Sections 7.3–7.4, which you might wish to read if you're curious. That proof is considerably more involved than the one we have given here; because of that, and because the area-based proof is perfectly rigorous and is much more likely to be found in high-school texts, we have chosen to stick with Euclid's approach. There is little to be gained by avoiding the use of area in the treatment of similar triangles.

There are many useful consequences of the Similar Triangles Theorem. The first one is really just a simple rephrasing of the theorem.

Corollary C.7. *If $\triangle ABC \sim \triangle DEF$, then there is a positive number r such that*

$$AB = r \cdot DE, \quad AC = r \cdot DF, \quad BC = r \cdot EF.$$

Proof. Just define r to be the ratio AB/DE, and use (C.4). □

Theorem C.8 (SAS Similarity Theorem). *If $\triangle ABC$ and $\triangle DEF$ are triangles such that $\angle A \cong \angle D$ and $AB/DE = AC/DF$, then $\triangle ABC \sim \triangle DEF$.*

Proof. Exercise C.1. □

Theorem C.9 (SSS Similarity Theorem). *If $\triangle ABC$ and $\triangle DEF$ are triangles such that $AB/DE = AC/DF = BC/EF$, then $\triangle ABC \sim \triangle DEF$.*

Proof. Exercise C.2. □

Note that the preceding theorem is actually the converse to the Similar Triangles Theorem. Taken together, these two theorems say that two triangles have equal corresponding angles if and only if they have proportional corresponding sides.

Theorem C.10 (Area Scaling Theorem). *If two triangles are similar, then the ratio of their areas is the square of the ratio of any two corresponding sides; that is, if $\triangle ABC \sim \triangle DEF$ and $AB = r \cdot DE$, then $\alpha(\triangle ABC) = r^2 \cdot \alpha(\triangle DEF)$.*

Proof. Exercise C.3. ☐

Finally, using the theory of similar triangles, we can give yet another proof of the Pythagorean Theorem. (Since it is one of the most significant theorems in all of mathematics, it never hurts to see another proof!) As usual, we will label the vertices of our right triangle A, B, and C, with the right angle at C; and we will denote the length of the leg opposite A by a, the length of the leg opposite B by b, and the length of the hypotenuse by c (Fig. 8).

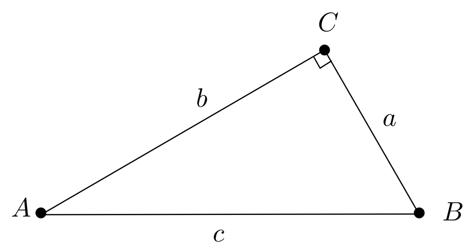

Figure 8: The Pythagorean Theorem: $a^2 + b^2 = c^2$.

Theorem C.11 (The Pythagorean Theorem). *Suppose $\triangle ABC$ is a right triangle with right angle $\angle C$, and let $a = BC$, $b = CA$, and $c = AB$. Then $a^2 + b^2 = c^2$.*

Proof. Let D be the foot of the perpendicular from C to $\overleftrightarrow{AB}$ (Fig. 9). Notice that D cannot coincide with

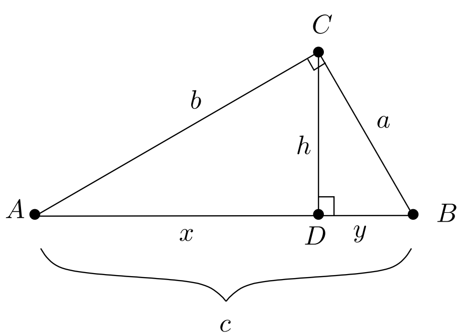

Figure 9: Proof of the Pythagorean Theorem.

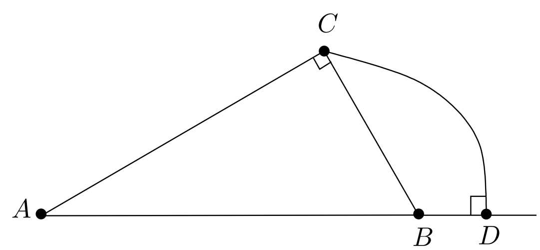

Figure 10: D cannot lie outside $\overline{AB}$.

A or B, for then $\triangle ABC$ would have a second right angle at A or B, contradicting Corollary B.9. Also, because $\angle A$ and $\angle B$ are acute, D cannot be outside of $\overline{AB}$ – for example, if $A * B * D$, then $\angle CBA$ would be an exterior angle for $\triangle CBD$ that is smaller than the remote interior angle at D (Fig. 10), contradicting the Exterior Angle Theorem. Thus D is an interior point of $\overline{AB}$. Set $x = AD$, $y = BD$, and $h = CD$.

By the Angle-Sum Theorem applied to the right triangles $\triangle CBD$ and $\triangle ABC$, $\mu\angle BCD + \mu\angle B = 90°$ and $\mu\angle A + \mu\angle B = 90°$. It follows by algebra that $\angle BCD \cong \angle A$, so by the AA Similarity Theorem, we conclude that $\triangle CBD \sim \triangle ABC$. The Similar Triangles Theorem then gives, among other things,

$$\frac{a}{c} = \frac{y}{a}.$$

The same argument also shows that $\triangle ACD \sim \triangle ABC$, so

$$\frac{b}{c} = \frac{x}{b}.$$

Simplifying both equations and adding them together, we obtain

$$\begin{aligned} a^2 &= cy, \\ b^2 &= cx, \\ a^2 + b^2 &= c(x+y). \end{aligned}$$

Since $x + y = c$, this proves the theorem. □

Our last theorem in this section is the last theorem in Euclid's Book I.

Theorem C.12 (Converse to the Pythagorean Theorem). *Suppose $\triangle ABC$ is a triangle, with side lengths $a = BC$, $b = CA$, and $c = AB$. If $a^2 + b^2 = c^2$, then $\angle C$ is a right angle.*

Proof. Exercise C.4. □

Exercises

C.1. Prove Theorem C.8 (the SAS Similarity Theorem). [Hint: If $AB < DE$, choose a point P in the interior of $\overline{DE}$ such that $DP = AB$, and let ℓ be the line through P parallel to $\overleftrightarrow{EF}$. Prove that ℓ intersects the interior of $\overline{DF}$ at a point Q, and use the hypothesis, the Side-Splitter Theorem, and some algebra to show that $\triangle ABC \cong \triangle DPQ$.]

C.2. Prove Theorem C.9 (the SSS Similarity Theorem). [Hint: The proof is very similar to that of the SAS Similarity Theorem.]

C.3. Prove Theorem C.10 (the Area Scaling Theorem). [Hint: Drop a perpendicular from a vertex to the opposite side in each triangle, and use the AA Similarity Theorem to show that this forms two pairs of similar triangles.]

C.4. Prove Theorem C.12 (the converse to the Pythagorean Theorem). [Hint: Construct a right triangle whose legs have lengths a and b, and show that it is congruent to $\triangle ABC$.]