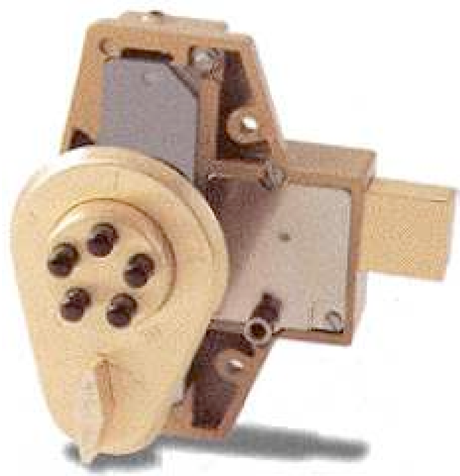
The Simplex Lock

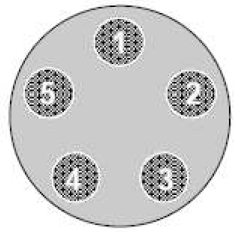
Here's a problem that will weave its way in and out of many of the topics we discuss this summer. We'll start working on it right now, but don't feel that you have to get the "right answer" right away. We'll keep coming back to it, even after you might be satisfied that you know what's going on. There's a lot to discover in this problem.

The Simplex company makes a combination lock that is used in many public buildings. It comes in several versions. Here are two:



These 5-button devices are purely mechanical (no electronics). You can set the combination using the following rules:

1. A combination is a sequence of 0 or more pushes, each push involving at least one button. (There's one possible no-push combination: the door's just already unlocked. Not a great combination, but it counts.)
2. Each button may be used at most once (once you press it, it stays in).
3. Each push may include any of the buttons that haven't been pushed yet, up to and including all remaining buttons.
4. The combination does *not* need to include all buttons.
5. When two or more buttons are pushed at the same time, order doesn't matter.



Artist's rendition of a Simplex lock.

Here are some possible combinations:

- $\{\{1,3\},\{4\}\}$
- $\{\{1,2\},\{3\}\}$
- $\{\{2\},\{1\},\{3\}\}$
- $\{\{1,2,4\},\{3,5\}\}$
- $\{\{1,2,3,4,5\}\}$
- $\{\{1,2\},\{4\},\{3,5\}\}$
- $\{\{3\},\{1,2\}\}$
- $\{\ \}$
- $\{2\}$

(Notation: $\{\{1,2\},\{3\}\}$ means "press 1 and 2 together, then press 3.") The company advertises thousands of combinations, and (as we say to our students) the question is, "Is the company telling the truth?"

How many combinations are there on a 5-button Simplex lock?

Session 1, continued

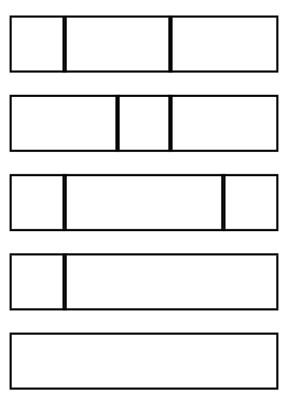
Welcome to *Take the Number Train!* In this course we will be working on topics in combinatorics, the art of counting. There will probably be ideas and notations you've seen before, and ones you have not.

Some notes on how the class is organized:

- Don't worry about answering all the questions. If you're answering every question, we haven't written the problem sets correctly.
- Don't worry about getting to a certain problem number. Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences). Also, skipping around is OK.
- Stop and smell the roses. Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you've encountered? How did others at your table think about this question?
- Teach only if you have to. You may feel the temptation to teach others at your table. Fight it! We don't mean you should ignore your tablemates but give everyone the chance to discover. If you think it's a good time to teach your tablemates about integration by parts, think again: the problems should lead to the appropriate mathematics rather than requiring it.

Okay, so let's get started. The first set of problems revolve around "trains" of rods. You can use rods of integer sizes to build "trains" that all share a common length. Graph paper is good for drawing these, and there are also Cuisenaire rods available for you make things hands-on.

A "train of length 5" is a row of rods whose combined length is 5. Here are some examples:



Notice that the 1-2-2 train and the 2-1-2 train contain the same rods but are listed separately. If you use identical rods in a different order, this is a separate train.

1. How many trains of length 5 are there? (Get out those Cuisenaire rods and make them! That's why they're here.)

2. Find a formula for the number of trains of length n. Come up with a convincing reason that your rule is correct.

3. Create an algorithm that will generate all the trains of length n.

4. How many trains of length n are there that use *only* cars of length 1 and 2? Find a general rule, and explain why your rule works.

5. How many trains of length 11 are there that use only cars of length 1, 2, and 3?

Tough Stuff

This section has some difficult problems! Try these if you're up for a challenge or already feel pretty confident about the problems in the rest of the set. Our guarantee: something challenging every day.

6. If you made all the trains of length 5, how many cars of length 1 were used? length 2, 3, 4, 5? See if you can find a general rule for the number of cars of length k you'd need to make all the trains of length n.

7. What's the *average* (mean) length of car used when you make all the trains of length 5? Is there a general rule at work here? Can you justify it?

Session 2: We All Scream

You'll notice that problem are numbered consecutively. It's a good idea to start each session with the new problems, but then feel free to go back to engaging problems from earlier sessions you want to spend more time on.

8. Suppose there are three flavors of ice cream: pistachio, strawberry, and chocolate. How many different three-scoop cones can you make using each of these flavors exactly once?

   Note that in a *cone*, it is important which scoop is on top. Thus, a pistachio-strawberry-chocolate cone is different from a strawberry-chocolate-pistachio cone.

9. Suppose you want a four-scoop cone with one scoop each of pistachio, strawberry, chocolate, and butter pecan. After your release from the mental hospital, how many different cones could you make with these flavors? Explain your reasoning carefully.

10. (a) How many different cones can you make from 5 scoops of different flavors?

    (b) How many different cones can you make from n scoops of different flavors?

11. Does problem **??** generally get easier or harder if you're allowed to repeat flavors within a cone? Why?

12. Day's Ice Cream serves 26 different flavors of ice cream.

    (a) How many different three-scoop cones can you make at Day's, if you never use the same flavor twice?

    (b) How many different four-scoop cones can you make at Day's, if you never use the same flavor twice?

    (c) Describe a rule for determining the number of different cones available at Day's in terms of the number of scoops on the cone.

13. In a *bowl* of ice cream, the order of the scoops does not matter. Therefore, a chocolate-vanilla bowl is the same as a vanilla-chocolate bowl.

    It's like the lock: pushing #4 and then #2 is like a cone; order matters. Pushing *both* #4 and #2 at once is like a bowl.

    (a) At a certain ice cream shop, you can make 465 different two-scoop bowls of ice cream (without repeating flavors). How many different two-scoop cones can you make? Explain how you know.

    (b) Find the number of flavors offered at this ice cream shop.

14. If you can make 220 different three-scoop bowls of ice cream, how many different three-scoop cones can you make? (Still no repeating flavors.)

15. (a) If you can make 210 different four-scoop bowls of ice cream, how many different four-scoop cones can you make?

    (b) If you can make 3024 different four-scoop *cones* of ice cream, how many different four-scoop *bowls* can you make?

16. If you can make 55440 different five-scoop cones, how many different five-scoop bowls can you make? (Please, a five-scoop bowl should only be ordered for mathematical purposes.)

17. (a) At Day's (where there are 26 flavors), how many different five-scoop bowls can be made?

    (b) Find a rule for determining the number of different bowls of k scoops at Day's.

18. Does problem **??** generally get easier or harder if you *a*re allowed to repeat flavors within a bowl? Why?

Tough Stuff

Remember that you can spend time on earlier Tough Stuff problems as well.

19. In a coin-flipping game, you flip a fair coin (heads or tails) ten times. If you flip heads twice in a row at any point during the game, you lose. Find the probability that you win at this game.

∞. Think about the Simplex lock problem again.